Elemente der Analysis II, Lösungen Übungsblatt 10

Ü46 (1) $\int_a^b \frac{1}{1+x^2}\,dx = \arctan\Big|_a^b = \arctan(b) - \arctan(a)$

(2) $\int_a^b \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_a^b \frac{2x}{1+x^2}\,dx = \frac{1}{2}\log(1+x^2)\Big|_{x=a}^b$

(3) $\int_a^b \frac{x^2}{1+x^2}\,dx = \int_a^b \frac{x^2+1-1}{1+x^2}\,dx = \int_a^b 1 - \frac{1}{1+x^2}\,dx$

$= (x - \arctan x)\Big|_{x=a}^b$

(4) $\int_a^b \frac{x^3}{1+x^2}\,dx = \int_a^b \frac{x^3+x-x}{1+x^2}\,dx = \int_a^b x - \frac{x}{1+x^2}\,dx$

$= \left(\frac{x^2}{2} - \frac{1}{2}\log(1+x^2)\right)\Big|_{x=a}^b$

Ü47 (1) $\int_1^e \frac{\log(x)}{x}\,dx = ?$

1. Lösung. Integrand von der Form $f'(x)f(x) = \left(\frac{1}{2}f^2(x)\right)'$

Also $\int_1^e \frac{\log(x)}{x}\,dx = \frac{1}{2}\log(x)^2\Big|_{x=1}^e = 1/2$

2. Lösung $\varphi(t) = e^t$, $1 = e^0$, $e = e^1$, $\varphi'(t) = e^t$

$\Rightarrow \int_1^e \frac{\log(x)}{x}\,dx = \int_0^1 \frac{\log(e^t)}{e^t}e^t\,dt = \int_0^1 t\,dt = t^2/2\Big|_0^1 = 1/2$

(2) $\int_1^e x\log(x)\,dx = ?$

1. Lösung partielle Integration $x = (x^2/2)'$, $\log(x)' = 1/x$

$\int_1^e x\log(x)\,dx = \frac{x^2}{2}\log(x)\Big|_{x=1}^e - \int_1^e \frac{x^2}{2}\cdot\frac{1}{x}\,dx = e^2/2 - \int_1^e \frac{x}{2}\,dx =$

$= e^2/2 - \frac{x^2}{4}\Big|_1^e = \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right) = e^2/4 - 1$

2. Lösung $x = e^t$, $dx = e^t dt$

$$\int_1^e x\log(x)\,dx = \int_0^1 e^t \log(e^t) e^t\,dt = \int_0^1 t e^{2t}\,dt \underset{\text{partiell}}{=}$$

$$= t\,e^{2t}/2\,\Big|_0^1 - \int_0^1 \frac{e^{2t}}{2}\,dt = e^2/2 - \left(\frac{e^{2t}}{4}\Big|_{t=0}^1\right) = \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right)$$

(3) $$\int_1^e (\log(x))^2 dx \underset{x=e^t}{=} \int_0^1 (\log(e^t))^2 e^t\,dt = \int_0^1 t^2 e^t\,dt =$$

$$= t^2 e^t\Big|_0^1 - \int_0^1 2t e^t\,dt = e - \left(2te^t\Big|_0^1 - \int_0^1 2e^t\,dt\right)$$

$$= e - \left(2e - \left(2e^t\Big|_0^1\right)\right) = e - (2e - (2e-2)) = e-2$$

(4) $$\int_1^{10} \frac{1}{\sqrt{x}(1+x)}\,dx \underset{\substack{x=t^2 \\ dx = 2t\,dt}}{=} \int_1^{\sqrt{10}} \frac{2t}{\sqrt{t^2}(1+t^2)}\,dt = \int_1^{\sqrt{10}} \frac{2}{1+t^2}\,dt$$

$$= 2\arctan\Big|_1^{\sqrt{10}} = 2\left(\arctan(\sqrt{10}) - \arctan(1)\right)$$

(5) $$\int_1^2 \sqrt{x}\log(x)\,dx \underset{x=e^t}{=} \int_0^{\log(2)} \sqrt{e^t}\,t\,e^t\,dt = \int_0^{\log(2)} e^{t/2} e^t\,t\,dt$$

$$= \int_0^{\log(2)} e^{\frac{3}{2}t}\,t\,dt = \frac{2}{3} e^{\frac{3}{2}t} t\Big|_0^{\log(2)} - \int_0^{\log(2)} \frac{2}{3} e^{\frac{3}{2}t}\,dt$$

$$= \frac{2}{3}\,2^{3/2}\log(2) - \frac{4}{9} e^{\frac{3}{2}t}\Big|_{t=0}^{\log(2)} = \frac{4}{3}\sqrt{2}\log(2) - \frac{4}{9}\,2^{3/2}\log(2) + \frac{4}{9}$$

Alternativ: partielle Integration $\sqrt{x} = x^{1/2} = \frac{2}{3}x^{3/2}$, $\log(x)' = 1/x$

$$\int_1^2 \sqrt{x}\log(x)\,dx = \frac{2}{3}x^{3/2}\log(x)\Big|_1^2 - \int_1^2 \frac{2}{3}x^{3/2}\frac{1}{x}\,dx = \frac{2}{3}2^{3/2}\log(2) - \frac{2}{3}\int_1^2 x$$

$$= \frac{4}{3}\sqrt{2}\log(2) - \frac{4}{9}\, x^{3/2}\Big|_1^2 = \frac{4}{3}\sqrt{2}\log(2) - \frac{8}{9}\sqrt{2}\log(2) - \frac{4}{9}$$

(6) $$\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\arcsin(1/2)} \frac{\sin(t)^2}{\sqrt{1-\sin^2(t)}} \cos(t)\,dt$$

$$= \int_0^{\arcsin(1/2)} \sin^2(t)\,dt = ?$$

Für $y > 0$ gilt mit partieller Integration

$$\int_0^y \sin^2(t)\,dt = \int_0^y \sin(t)\sin(t)\,dt = -\cos(y)\sin(y) + \int_0^y \cos(t)\cos(t)\,dt$$

$$= -\cos(y)\sin(y) + \int_0^y 1 - \sin^2(t)\,dt$$

$$= -\cos(y)\sin(y) + y - \int_0^y \sin^2(t)\,dt$$

$$\Rightarrow \int_0^y \sin^2(t)\,dt = \frac{1}{2}\left(y - \cos(y)\sin(y)\right)$$

$$\Rightarrow \int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\left(\arcsin(1/2) - \cos(\arcsin(1/2))\,\tfrac{1}{2}\right)$$

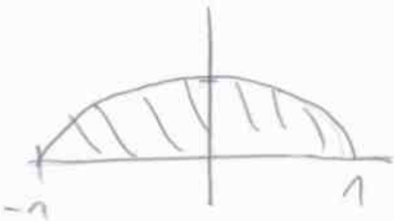
$$\underset{\cos = \sqrt{1-\sin^2}}{=} \frac{1}{2}\left(\arcsin(1/2) - \sqrt{1-1/2}\ \tfrac{1}{2}\right) = \frac{1}{2}\arcsin(1/2) - \frac{1}{4\sqrt{2}}$$

($\arcsin(1/2) = \pi/6$, aber das müsste man in der Klausur nicht wissen, abgesehen davon, dass dieses Integral für die Klausur viel zu kompliziert wäre.)

Ü48 (a)

$$\text{Fläche} = \int_1^e \log(x)\,dx = x\log(x) - x \Big|_{x=1}^{e} = e\log(e) - e - (1\log(1) - 1)$$
$$= 1$$

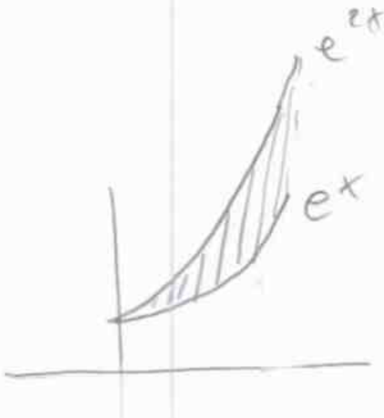
(b) B $\qquad B = \{ \begin{bmatrix} x \\ y \end{bmatrix} : -1 \leq x \leq 1,\ 0 \leq y \leq \frac{1}{2}\sqrt{1-x^2} \}$



halbe Ellipse

$$\text{Fläche} = \int_{-1}^{1} \frac{1}{2}\sqrt{1-x^2}\,dx = \frac{1}{2}\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{1}{2}$$ Fläche eines Halbkreises mit Radius 1 $= \frac{1}{2}\frac{\pi}{2} = \pi/4$.

(c) Eigentlich noch nicht definiert, aber



$$\text{Fläche} = \int_0^1 e^{2x}\,dx - \int_0^1 e^x\,dx$$
$$= \frac{e^{2x}}{2}\Big|_0^1 - e^x\Big|_0^1 = \frac{e^2}{2} - 1 - (e-1) = e^2/2 - e = (e/2 - 1)e$$

Ü49 (a) TRICK $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$ hat Stammfunktion

$$F(x) = \log(x) - \log(x+1) \quad \left(= \log\left(\frac{x}{x+1}\right)\right)$$

(b) $F(x) = \int_0^x \frac{t}{1+t^4}\,dt = \frac{1}{2}\int_0^x \frac{2t}{1+(t^2)^2}\,dt = \frac{1}{2}\arctan(x^2)$

(c) $f = \cos$ hat Stammfunktion $F(x) = \sin(x) = \sqrt{1-\cos^2(x)}$ ($\sin \geq 0$ auf $[0,\pi]$)

$\Rightarrow$ arccos hat Stammfunktion $G(y) = y\arccos(y) - F(f^{-1}(y))$

$= y\arccos(y) - \sqrt{1-y^2}$