# On Hipp's compound Poisson approximations via concentration functions

BERO ROOS

*Fachbereich Mathematik, SPST, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany. E-mail: roos@math.uni-hamburg.de*

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**Abstract:** This paper is devoted to a refinement of Hipp's method in the compound Poisson approximation to the distribution of the sum of independent but not necessarily identically distributed random variables. Approximations by related Kornya-Presman signed measures are also considered. By using alternative proofs, we show that several constants in the upper bounds for the Kolmogorov and the stop-loss distances can be reduced. Concentration functions play an important rôle in Hipp's method. Therefore, we provide an improvement of the constant in Le Cam's bound for concentration functions of compound Poisson distributions. But we also follow Hipp's idea to estimate such concentration functions with the help of Kesten's concentration function bound for sums of independent random variables. In fact, under the assumption that the summands are identically distributed, we present a smaller constant in Kesten's bound, the proof of which is based on a slight sharpening of Le Cam's version of the Kolmogorov-Rogozin inequality.

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# 1 Introduction

## 1.1 Motivation

The compound Poisson approximation of the distribution of the sum of independent but not necessarily identically distributed random variables has a long history. Such an approximation is reasonable when the summands are nonzero with small probabilities. In fact, in this case, the approximation error between the involved distributions is small. Though several upper bounds for different distances are nowadays available, there remain some difficult tasks. For instance, there is the problem to give a good estimate for the constant in the upper bound by Zaĭtsev (1983, formula (6), p. 658); in fact, he improved the order of Le Cam's (1965, Theorem 3, p. 188; see also 1986, Proposition 4, pp. 413–414) bound by using the so called "method of triangular functions", which was invented by Arak and Zaĭtsev in the 1980s in order to find the optimal rate in Kolmogorov's (1956) second uniform limit theorem. For details of this method, see the monograph by Arak and Zaĭtsev (1988). Further developments can be found, for example, in Čekanavičius (2003) and his previous papers. Because of the complexity of this method, constants are given not in detail; even if one follows the proofs by taking into account explicit constants, the final constant would be very large. In order to avoid this difficulty, Hipp (1985; 1986) invented his own method and proved some estimates, which are not easily comparable with the Zaĭtsev bound. As approximations, he not only considered the compound Poisson distribution but also finite signed measures, which can be derived from an expansion in the exponent. Apparently, such approximations were first considered by Kornya (1983) and Presman (1983), wherefore we speak of Kornya-Presman signed measures. However, observe that the signed measures used by Kornya and Presman are slightly different (see also Hipp 1986). Further results in this direction came, for example, from Kruopis (1986), Čekanavičius (1997), Barbour and Xia (1999), and Roos (2002). Note that Barbour, Chen, and Loh (1992) and Barbour and Xia (1999; 2000) applied Stein's method but obtained some unwanted terms in their bounds, some of which could be removed by using Kerstan's approach (see Roos 2003). However, it should be mentioned that, in contrast to Kerstan's approach, Stein's method also works in the context of dependent variables. In the latter paper, a more detailed review of known results can be found.

This paper is devoted to a refinement of Hipp's (1985; 1986) method. Motivated by the need for explicit approximation results for the individual aggregate claims distribution within the context of risk theory, Hipp has used concentration functions in his estimates of the Kolmogorov and stop-loss distances. The aim of the present paper is to present alternative proofs and smaller constants in the bounds. Additionally, we provide an improvement of the constant in Le Cam's (1986, Remark on page 408) bound for concentration functions of compound Poisson distributions. But we also follow Hipp's idea to give an estimate of such concentration functions with the help of Kesten's (1969) concentration function bound for sums of independent random variables. In fact, under the assumption that the summands are identically distributed, we present a smaller constant in Kesten's bound, the proof of which is based on a slight sharpening of Le Cam's (1986, Theorem 2, p. 411) version of the Kolmogorov-Rogozin inequality; see Kolmogorov (1958) and Rogozin (1961, Theorem 1, p. 95). For the theory of concentration functions, the reader is referred to Hengartner and Theodorescu (1973), Petrov (1975; 1995), and Arak and Zaĭtsev (1988). Note that, in the literature, many contributions on upper bounds of concentration functions can be found. But only a small number of them deal with explicit constants; for instance, see the recent papers by Salikhov (1996) and Nagaev and Khodzhibagyan (1996). Though it is often possible to derive bounds for concentration functions for sums of independent but not necessarily identically distributed random variables, in this paper, we only need to consider identically distributed summands.

## 1.2 Some notation

### 1.2.1 Concentration functions

The concentration functions $\mathrm{Conc}(Q\,;\,\cdot)$, $\mathrm{Conc}^-(Q\,;\,\cdot) : [0,\infty) \to [0,1]$ of a probability measure $Q$ on $\mathbb{R}$ are defined by

$$\begin{aligned} \mathrm{Conc}(Q\,;\,t) &= \sup_{x\in\mathbb{R}} Q([x, x+t]), \\ \mathrm{Conc}^-(Q\,;\,t) &= \sup_{x\in\mathbb{R}} Q((x, x+t]), \qquad t \in [0,\infty). \end{aligned}$$

In the appendix (see Section 4), we have listed some basic properties of concentration functions.

### 1.2.2 Stop-loss transforms

For a finite signed measure $Q$ on $\mathbb{R}$, let $|Q|$ denote the total variation measure and $F_Q = Q((-\infty, \cdot])$ the distribution function of $Q$. The stop-loss transform $\pi_Q$ of $Q$ at point $t \in \mathbb{R}$ is defined by

$$\pi_Q(t) = \int_{\mathbb{R}} (x-t)_+ \, \mathrm{d}Q(x),$$

Here and throughout this paper, $x_+ = x \vee 0$, $x \vee y = \max\{x, y\}$, and $x \wedge y = \min\{x, y\}$ for $x, y \in \mathbb{R}$. Whenever we deal with a stop-loss transform $\pi_Q$, to ensure that $\pi_Q$ has finite values, we assume that $\int_{\mathbb{R}} |x| \, \mathrm{d}|Q|(x) < \infty$. For a real-valued random variable $X$ with distribution $\mathcal{L}(X)$ and distribution function $F_X = F_{\mathcal{L}(X)}$, we set $\overline{F}_X = 1 - F_X$; if $\mathrm{E}(X)$ is finite, the stop-loss transform $\pi_X = \pi_{\mathcal{L}(X)}$ of $X$ is finite and satisfies, for $t \in \mathbb{R}$,

$$\pi_X(t) = \mathrm{E}(X-t)_+ = \int_t^\infty \overline{F}_X(x) \, \mathrm{d}x = \mathrm{E}(X_+) - \int_0^t \overline{F}_X(x) \, \mathrm{d}x,$$

where, as usual, "$\int_x^y = -\int_y^x$" for $x, y \in \mathbb{R}$. Similar formulas for $\pi_Q$ are possible, when $Q$ is a finite signed measure. Note that, in the context of stop-loss reinsurance, a risk $X$ (i.e. a non-negative random variable) is divided between the ceding company and the reinsurer in such a way that the reinsurer has to pay the excess $(X-t)_+$ over an agreed retention $t > 0$, whereas the ceding company has to pay the remaining amount $X \wedge t$; hence $\pi_X(t)$ denotes the expected claim of the reinsurer.

### 1.2.3 Distances

As measures of accuracy, we consider the following distances

$$\begin{aligned} d_{\mathrm{KM}}(Q_1, Q_2) &= \sup_{x \in \mathbb{R}} |F_{Q_1}(x) - F_{Q_2}(x)|, & \text{(Kolmogorov metric)}, \\ d_{\mathrm{SL}}(Q_1, Q_2) &= \sup_{t \in \mathbb{R}} |\pi_{Q_1}(t) - \pi_{Q_2}(t)|, & \text{(stop-loss metric)}, \end{aligned}$$

between two finite signed measures $Q_1$ and $Q_2$ on $\mathbb{R}$. For two real-valued random variables $X$ and $Y$, we write

$$d_{\mathrm{KM}}(X, Y) = d_{\mathrm{KM}}(\mathcal{L}(X), \mathcal{L}(Y)) \quad \text{and} \quad d_{\mathrm{SL}}(X, Y) = d_{\mathrm{SL}}(\mathcal{L}(X), \mathcal{L}(Y)).$$

Sometimes it will be necessary to consider also the Fortet-Mourier metric

$$d_{\mathrm{FM}}(X, Y) = d_{\mathrm{FM}}(\mathcal{L}(X), \mathcal{L}(Y)) = \int_{\mathbb{R}} |F_X(x) - F_Y(x)| \, \mathrm{d}x$$

between $X$ and $Y$ and an $\ell_1$-version of the stop-loss metric

$$\tilde{d}_{\mathrm{SL}}(M,\, N) = \tilde{d}_{\mathrm{SL}}(\mathcal{L}(M),\, \mathcal{L}(N)) = \sum_{n=0}^{\infty} |\pi_M(n) - \pi_N(n)|,$$

between random variables $M$ and $N$ concentrated on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

### 1.2.4 Exponentials

In what follows, we need exponentials of finite signed measures. If $Q$ denotes a finite signed measure on $\mathbb{R}$, then we set

$$\exp(Q) = \sum_{j=0}^{\infty} \frac{1}{j!} Q^{*j},$$

where, for $j \in \mathbb{N} = \{1, 2, \ldots\}$, $Q^{*j}$ denotes the $j$-fold convolution of $Q$ with itself and $Q^{*0} = \varepsilon_0$ is the Dirac measure at point zero. Note that $\exp(Q)$ is a finite signed measure. Is is well-known that, for finite signed measures $Q_1$ and $Q_2$, we have $\exp(Q_1) * \exp(Q_2) = \exp(Q_1 + Q_2)$; for example, see Hipp (1985; 1986) and Hipp and Michel (1990, Kapitel 4). A proof of this and other similar facts regarding finite signed measures can easily be done with the help of the Hahn-Jordan decomposition and characteristic functions. For a probability distribution $Q$ on $\mathbb{R}$ and parameter $t \in [0, \infty)$, we define the compound Poisson distribution by

$$\mathrm{CPo}(t,\, Q) = \exp(t(Q - \varepsilon_0)) = \sum_{j=0}^{\infty} \mathrm{po}(j,\, t)\, Q^{*j},$$

where $\mathrm{po}(j,\, t) = \mathrm{e}^{-t}\, t^j / j!$.

# 2 Results

## 2.1 Hipp-type results

In the following proposition, we are concerned with the approximation by a compound Poisson distribution.

**Proposition 1** *Let $n \in \mathbb{N}$ and $X_1, \ldots, X_n$ be non-negative and independent random variables. Set $S_n = \sum_{i=1}^n X_i$ and, for all $i \in \{1, \ldots, n\}$,*

$$\begin{aligned} p_i &= P(X_i > 0), \qquad Q_i = P(X_i \in \cdot \,|\, X_i > 0), \\ \mu_i &= \int x \,\mathrm{d}Q_i(x), \qquad \mu_i^{(2)} = \int x^2 \,\mathrm{d}Q_i(x), \end{aligned}$$

$$c_i = \begin{cases} 1, & \textit{if } Q_i \textit{ is a Dirac measure,} \\ 2, & \textit{otherwise,} \end{cases} \qquad c_i' = \frac{c_i - 1}{4}.$$

*Let*

$$\begin{aligned} \lambda &= \sum_{i=1}^n p_i, \qquad Q = \frac{1}{\lambda}\sum_{i=1}^n p_i\, Q_i, \qquad H = \mathrm{CPo}(\lambda,\, Q), \\ \tilde\lambda &= \frac{1}{2}\sum_{i=1}^n p_i(1-p_i), \qquad \tilde Q = \frac{1}{\tilde\lambda}\sum_{i=1}^n \frac{p_i(1-p_i)}{2} Q_i, \qquad \tilde H = \mathrm{CPo}(\tilde\lambda,\, \tilde Q). \end{aligned}$$

*If, for all $i$, $p_i < 1$, then*

$$\begin{aligned} d_{\mathrm{KM}}(\mathcal{L}(S_n),\, H) &\le \frac{\pi^2}{8}\sum_{i=1}^n \frac{c_i\, p_i^2}{1-p_i} \,\mathrm{Conc}^-(\tilde H\,;\, \mu_i), &(1)\\ d_{\mathrm{SL}}(\mathcal{L}(S_n),\, H) &\le \frac{\pi^2}{4}\sum_{i=1}^n \frac{p_i^2}{1-p_i}\Big(\mu_i + c_i'\Big(\mu_i + \frac{\mu_i^{(2)}}{\mu_i}\Big)\Big)\mathrm{Conc}^-(\tilde H\,;\, \mu_i). &(2) \end{aligned}$$

**Remark 1** (a) Inequality (1) can essentially be found in Hipp and Michel (1990, p. 51); see also Hipp (1985). Exactly the same result can also be derived by using the proof by Hipp and Michel. The bound (2) is slightly sharper than the one in Hipp and Michel (1990, p. 54). In fact, for non-degenerate probability distribution $Q_i$, their bound contains the term $\mu_i + \mu_i^{(2)}/(2\mu_i)$ instead of the smaller value

$$\mu_i + \frac{1}{4}\Big(\mu_i + \frac{\mu_i^{(2)}}{\mu_i}\Big).$$

(b) It is well-known that, under the assumptions of Proposition 1, the distribution $\mathcal{L}(S_n)$ is smaller or equal to $H$ in the stop-loss order, i.e., for all $t \in \mathbb{R}$, we have $\pi_{S_n}(t) \le \pi_H(t)$; see Hipp and Michel (1990, p. 43). This maybe helpful when dealing with the stop-loss distance.

(c) As pointed out by Hipp (1985), in order to obtain higher accuracy, the concentration functions in the upper bounds of Proposition 1 should be evaluated rather than estimated. Indeed, in many applications, where $\tilde Q$ is an arithmetic probability distribution with $\tilde Q(\{h,\, 2h,\, 3h,\, \ldots\}) = 1$ and $h \in (0, \infty)$, $\mathrm{Conc}^-(\tilde H\,;\, \mu_i)$ can be evaluated by using Panjer's (1981) recursive algorithm. Nevertheless, we provide some general upper bounds for concentration functions in Section 2.3.

(d) Note that Zaĭtsev (1983, formula (6), p. 658) has shown that

$$d_{\mathrm{KM}}(\mathcal{L}(S_n),\, H) \le c \max_{1\le i\le n} p_i,$$

where $c$ denotes an absolute constant.

**Remark 2** The situation in Proposition 1 can be interpreted within risk theory: let us consider the individual model with a portfolio of $n \in \mathbb{N}$ independent policies, producing the non-negative individual claim amounts $X_1, \dots, X_n$. Each $X_i$ can be written as a random sum $X_i = \sum_{k=1}^{M_i} U_{i,k}$. Here, for $i$ being fixed, the $U_{i,k}$ for $k \in \mathbb{N}$ are positive, independent and identically distributed random variables and $M_i$ is a Bernoulli random variable independent of the $U_{i,k}$ with $P(M_i = 1) = 1 - P(M_i = 0) = p_i$. The $p_i$ represents the probability that risk $i$ produces a positive claim, wherefore we can assume that $p_i$ is small. Further, $\mathcal{L}(U_{i,1}) = Q_i$ is the conditional distribution of the claim in risk $i$, given that a positive claim occurs in risk $i$. The aggregate claim in the individual model is defined by the sum $S_n$ of all $X_i$. Frequently the distribution $\mathcal{L}(S_n)$ of $S_n$ is quite involved and should be approximated by a simpler distribution. Due to the smallness of the $p_i$, an approximation by a compound Poisson distribution $\mathrm{CPo}(\lambda,\, Q)$ is particularly favourable. Note that we obtain this distribution by Poissonization: if, in the sum $X_i = \sum_{k=1}^{M_i} U_{i,k}$, we replace $M_i$ with an independent Poisson distributed random variable $N_i$ with the same mean as $M_i$, then we obtain random variables $Y_i = \sum_{k=1}^{N_i} U_{i,k}$. Now, $\mathcal{L}(\sum_{i=1}^{n} Y_i) = \mathrm{CPo}(\lambda,\, Q)$. From this, we see that Corollary 1 below is applicable. The distribution $\mathrm{CPo}(\lambda,\, Q)$ can also be obtained as the aggregate claims distribution $\sum_{j=1}^{\tilde{M}} V_j$ of a suitable collective model: here only the claims $V_j$ and their total number $\tilde{M}$ are modeled. In the present context, the claim number $\tilde{M}$ has a Poisson $\mathrm{Po}(\lambda)$ distribution with mean $\lambda$ and the claims are independent (also of the claim number) and identically distributed random variables with distribution $Q$.

The next proposition deals with the approximation by Kornya-Presman signed measures $H_K$ (see below).

**Proposition 2** *Let the assumptions of Proposition 1 be valid. Further, set $K \in \mathbb{N}$,*

$$H_K = \exp\Big(\sum_{i=1}^{n}\sum_{k=1}^{K} \frac{(-1)^{k+1}}{k} p_i^k (Q_i - \varepsilon_0)^{*k}\Big),$$

$$\tau(i, K) = \frac{(2p_i)^{K+1}}{(K+1)(1-2p_i)} \quad \textit{for } i \in \{1, \dots, n\},$$

$$\delta = \sum_{i=1}^{n} (\mathrm{e}^{\tau(i,K)} - 1), \qquad \eta = \frac{\pi^2}{8(1-\delta)},$$

*where we assume that, for all $i$, $p_i < 1/2$ and $\delta < 1$. Then*

$$d_{\mathrm{KM}}(\mathcal{L}(S_n),\, H_K) \;\leq\; \eta \sum_{i=1}^{n} c_i (\mathrm{e}^{\tau(i,K)} - 1)\, \mathrm{Conc}^{-}(\tilde{H}\,;\, \mu_i), \tag{3}$$

$$d_{\mathrm{SL}}(\mathcal{L}(S_n),\, H_K) \;\leq\; \eta \sum_{i=1}^{n} (\mathrm{e}^{\tau(i,K)} - 1) \Big(\mu_i + c_i' \Big(\mu_i + \frac{\mu_i^{(2)}}{\mu_i}\Big)\Big) \mathrm{Conc}^{-}(\tilde{H}\,;\, \mu_i). \tag{4}$$

**Remark 3** Inequality (3) is better than the one by Hipp and Michel (1990, p. 82); see also Hipp (1986). In fact, their bound contains the values $\mathrm{e}^{2\tau(i,K)}$ and $\mathrm{Conc}^{-}(\tilde{H}\,;\,(K+1)\mu_i)$ instead of the better ones $\mathrm{e}^{\tau(i,K)}$ and $\mathrm{Conc}^{-}(\tilde{H}\,;\,\mu_i)$. For $d_{\mathrm{SL}}(\mathcal{L}(S_n),\, H_K)$, we found no comparable bounds in the literature; therefore (4) seems to be new. However, in Hipp (1986, formula (10)), a non-uniform inequality for the difference of the stop-loss transforms of $\mathcal{L}(S_n)$ and the signed measures originally used by Kornya (1983) was presented. Note, as mentioned above, these signed measures differ slightly from the $H_K$ of the present paper.

**Remark 4** The idea behind the use of the finite signed measure $H_K$ is the following: using the log-series and characteristic functions, it is easy to show that, for $i \in \{1, \ldots, n\}$,

$$\mathcal{L}(X_i) = \varepsilon_0 + p_i(Q_i - \varepsilon_0) = \exp\Big(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} p_i^k (Q_i - \varepsilon_0)^{*k}\Big).$$

Note that, since $p_i < 1/2$, the infinite sum in the exponent converges with respect to the total variation norm and forms a finite signed measure. We obtain

$$\mathcal{L}(S_n) = \exp\Big(\sum_{i=1}^{n}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} p_i^k (Q_i - \varepsilon_0)^{*k}\Big);$$

see also Hipp and Michel (1990, Kapitel 4). Therefore, we should expect that $H_K$ is a good approximation of $\mathcal{L}(S_n)$, if $K$ is large. In fact, from Proposition 2, it follows that $d_{\mathrm{KM}}(\mathcal{L}(S_n),\, H_K)$ and $d_{\mathrm{SL}}(\mathcal{L}(S_n),\, H_K)$ tend to zero as $K \to \infty$, if $p_i < 1/2$ and if the respective moments of $Q_i$ are finite for all $i$.

## 2.2 The main tool for Proposition 1

The following theorem seems to be new.

**Theorem 1** *Let $X_1, X_2, X_3, \dots$ be non-negative, independent and identically distributed random variables. For $n \in \mathbb{Z}_+$, set $S_n = \sum_{i=1}^n X_i$. Let $M$ and $N$ be $\mathbb{Z}_+$-valued random variables with the same finite expectation. Let $Y$ denote a random variable in $\mathbb{R}$. We assume that all $Y, M, N, X_1, X_2, \dots$ are independent.*

(a) *Then we have*

$$d_{\mathrm{KM}}(S_M + Y,\, S_N + Y) \leq \frac{1}{2}\, d_{\mathrm{FM}}(M,\, N)\, d_{\mathrm{KM}}(Y,\, X_1 + Y).$$

(b) *If* $\mathrm{E}(X_1) < \infty$*, then*

$$d_{\mathrm{SL}}(S_M + Y,\, S_N + Y) \leq \tilde{d}_{\mathrm{SL}}(M,\, N)\, \mathrm{E}[(X_1 \wedge X_2)\, \mathrm{Conc}^-(\mathcal{L}(Y)\,;\, X_1 + X_2)].$$

Note that the upper bounds in Theorem 1 are small, when $\mathcal{L}(M) \approx \mathcal{L}(N)$, or $\mathcal{L}(X_1) \approx \varepsilon_0$, or when $\mathcal{L}(Y)$ has a small concentration. Theorem 1 and the telescopic sum decomposition enable us to give results concerning the approximation of sums of independent but not necessarily identically distributed random variables.

**Corollary 1** *Let $n \in \mathbb{N}$ and $X_1, \dots, X_n, Y_1, \dots, Y_n$ be independent random variables. For each $i \in \{1, \dots, n\}$, $X_i$ and $Y_i$ are given by random sums of the form $X_i = \sum_{k=1}^{M_i} U_{i,k}$ and $Y_i = \sum_{k=1}^{N_i} U_{i,k}$, where, for $i$ being fixed, the $U_{i,1}, U_{i,2}, U_{i,3}, \dots$ are non-negative, independent and identically distributed random variables and the $M_i$ and $N_i$ are random variables in $\mathbb{Z}_+$ with $\mathrm{E}(M_i) = \mathrm{E}(N_i) < \infty$. We assume that all $M_i, N_i, U_{i,k}$ are independent. Set $S_n = \sum_{i=1}^n X_i$, $T_n = \sum_{i=1}^n Y_i$ and, for $i \in \{1, \dots, n\}$, $Z_i = \sum_{j=1}^{i-1} X_j + \sum_{j=i+1}^n Y_j$.*

(a) *Then we have*

$$d_{\mathrm{KM}}(S_n,\, T_n) \leq \frac{1}{2} \sum_{i=1}^n d_{\mathrm{FM}}(M_i,\, N_i)\, d_{\mathrm{KM}}(Z_i,\, U_{i,1} + Z_i).$$

(b) *If* $\mathrm{E}(U_{i,1}) < \infty$ *for all* $i \in \{1, \dots, n\}$*, then*

$$d_{\mathrm{SL}}(S_n,\, T_n) \leq \sum_{i=1}^n \tilde{d}_{\mathrm{SL}}(M_i,\, N_i)\, \mathrm{E}[(U_{i,1} \wedge U_{i,2})\, \mathrm{Conc}^-(\mathcal{L}(Z_i)\,;\, U_{i,1} + U_{i,2})].$$

**Proof.** The assertion easily follows from Theorem 1 in conjunction with the well-known telescopic sum decomposition

$$\mathcal{L}(S_n) - \mathcal{L}(T_n) = \sum_{i=1}^n (\mathcal{L}(X_i + Z_i) - \mathcal{L}(Y_i + Z_i)),$$

which, in turn, can be shown via induction over $n$. $\square$

The preceding corollary is used to give the proof of Proposition 1.

## 2.3 Concentration function bounds

The following proposition is devoted to Le Cam-type bounds for the concentration functions of a compound Poisson distribution. The absolute constant $(2\mathrm{e})^{-1/2}$ in the bounds is best possible.

**Proposition 3** *Let $Q$ be a probability distribution on $\mathbb{R}$. Then, for $t \in (0, \infty)$ and $s \in [0, \infty)$,*

$$\mathrm{Conc}(\mathrm{CPo}(t, Q)\,;\, s) \;\leq\; \frac{1}{\sqrt{2\mathrm{e}\, t\, f(s, Q)}}, \tag{5}$$

$$\mathrm{Conc}^{-}(\mathrm{CPo}(t, Q)\,;\, s) \;\leq\; \frac{1}{\sqrt{2\mathrm{e}\, t\, g(s, Q)}}, \tag{6}$$

*where*

$$\begin{aligned} f(s, Q) &= \max\{Q((-\infty, -s)),\, Q((s, \infty))\}, \\ g(s, Q) &= \max\{Q((-\infty, -s]),\, Q([s, \infty))\}. \end{aligned}$$

*In (5) and (6), equalities hold, when $s \in (0, 1)$, $t = 1/2$, and $Q = \varepsilon_1$ is the Dirac measure at point one such that $\mathrm{CPo}(t, Q) = \mathrm{Po}(1/2)$.*

**Remark 5** From Le Cam's (1986, Remark on p. 408) more general inequality for the concentration function of an infinitely divisible probability distribution, it follows that, under the assumptions of Proposition 3,

$$\mathrm{Conc}(\mathrm{CPo}(t, Q)\,;\, s) \leq \Big(\frac{2\pi}{t\, Q(\{x :\, |x| > s\})}\Big)^{1/2}; \tag{7}$$

(see also Le Cam 1965, Proposition 5, p. 183; Arak and Zaĭtsev 1988, Theorems 2.5 and 2.6, p. 46). Since $f(s, Q) \geq 2^{-1} Q(\{x :\, |x| > s\})$, it follows from (5) that the constant $\sqrt{2\pi} \approx 2.51$ in (7) can be replaced with $\mathrm{e}^{-1/2} \approx 0.61$.

The bound (6) can be used to estimate the concentration functions in the upper bounds in Propositions 1 and 2. However, other bounds can be derived with the help of a Kesten-type inequality for the concentration function of the sum of independent and identically distributed random variables:

**Proposition 4** *Let $S_n = \sum_{i=1}^{n} X_i$ be the sum of $n \in \mathbb{N}$ independent and identically distributed random variables $X_1, \ldots, X_n$. Then, for $t \in [0, \infty)$,*

$$\mathrm{Conc}(\mathcal{L}(S_n)\,;\, t) \leq 6.33 \frac{\mathrm{Conc}(\mathcal{L}(X_1)\,;\, t)}{\sqrt{(n+1)(1 - \mathrm{Conc}(\mathcal{L}(X_1)\,;\, t))}}. \tag{8}$$

*This inequality remains valid if* Conc *is everywhere replaced with* $\mathrm{Conc}^{-}$.

**Remark 6** From a more general result by Kesten (1969, Corollary 1, pp. 134–135), it follows that, under the assumptions of Proposition 4,

$$\mathrm{Conc}(\mathcal{L}(S_n)\,;\,t) \leq 4\sqrt{2}(1+9c)\frac{\mathrm{Conc}(\mathcal{L}(X_1)\,;\,t)}{\sqrt{n(1-\mathrm{Conc}(\mathcal{L}(X_1)\,;\,t))}}. \tag{9}$$

Here, $c$ is an absolute constant satisfying the classical Kolmogorov-Rogozin inequality (see Kolmogorov 1958; Rogozin 1961, Theorem 1, p. 95), which states that, under the same assumptions,

$$\mathrm{Conc}(\mathcal{L}(S_n)\,;\,t) \leq \frac{c}{\sqrt{n(1-\mathrm{Conc}(\mathcal{L}(X_1)\,;\,t))}}. \tag{10}$$

Since $c \leq 1$ (see Remark 8 below), the leading constant in (9) is bounded from above by $40\sqrt{2} \leq 56.6$, which is considerably larger than our 6.33. A further advantage of (8) over (9) is the factor $(n+1)^{-1/2}$ instead of $n^{-1/2}$.

**Corollary 2** *Under the assumptions of Proposition 3, we have*

$$\mathrm{Conc}(\mathrm{CPo}(t,\,Q)\,;\,s) \leq \mathrm{e}^{-t} + \frac{6.33\,\mathrm{Conc}(Q\,;\,s)(1-\mathrm{e}^{-t})}{\sqrt{t\,(1-\mathrm{Conc}(Q\,;\,s))}}. \tag{11}$$

*This inequality remains valid if* Conc *is everywhere replaced with* $\mathrm{Conc}^-$.

**Remark 7** (a) The bound (11) can be much better than (5) if $t$ is large and if $Q$ has a small concentration function.

(b) Bening et al. (1997, Theorem 8, pp. 370–371) have shown that, under the assumptions of Proposition 3,

$$\mathrm{Conc}(\mathrm{CPo}(t,\,Q)\,;\,s) \leq c(\epsilon,\,\delta)\,\frac{s+1}{\sqrt{t}}, \tag{12}$$

where

$$c(\epsilon,\,\delta) = \left(\frac{96}{95}\right)^2 \max\left\{1,\,\frac{1}{\delta}\right\}\sqrt{\frac{\pi}{\epsilon}}$$

and $\epsilon, \delta > 0$ are defined in such a way that the characteristic function $\varphi_Q(x) = \int \mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}Q(y)$ of $Q$ satisfies

$$|\varphi_Q(x)| \leq 1 - \epsilon x^2 \qquad \text{whenever } |x| \leq \delta.$$

In fact, for each non-degenerate probability distribution $Q$, there exist positive numbers $\epsilon$ and $\delta$ with such a property (see Petrov 1975, Theorem 1.2.2, p. 11). It is easily shown that, under the present assumptions, $\max\{1, \delta^{-1}\}\epsilon^{-1/2} \geq 1$. Therefore the bound in (12) is often worse than the one in (11).

The proof of Proposition 4 is based on a refinement of Le Cam's version of the Kolmogorov-Rogozin inequality for the concentration function of the sum of independent random variables.

**Proposition 5** *Under the assumptions of Proposition 4, we have*

$$\mathrm{Conc}(\mathcal{L}(S_n)\,;\,t) \leq \Big(\frac{1-[\mathrm{Conc}(\mathcal{L}(X_1)\,;\,t)]^{n+1}}{(n+1)(1-\mathrm{Conc}(\mathcal{L}(X_1)\,;\,t))}\Big)^{1/2}.$$

*This inequality remains valid if* Conc *is everywhere replaced with* $\mathrm{Conc}^-$.

**Remark 8** From the more general Theorem 2 in Le Cam (1986, p. 411), it follows that, in the Kolmogorov-Rogozin inequality (see (10)), one can choose $c = 1$. But his inequality is even a bit better. In fact, under the assumptions of Proposition 4, his result implies that

$$\mathrm{Conc}(\mathcal{L}(S_n)\,;\,t) \leq \Big(\frac{1-\exp(-n(1-\mathrm{Conc}(\mathcal{L}(X_1)\,;\,t)))}{n(1-\mathrm{Conc}(\mathcal{L}(X_1)\,;\,t))}\Big)^{1/2}.$$

However, it is easily shown that this bound is always larger than or equal to the one of Proposition 5.

# 3 Remaining proofs

In what follows, we use the forward difference operator $\Delta b:\ \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$ of a sequence $b: \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$, which is defined by $\Delta b_n = b_n - b_{n+1}$ for $n \in \mathbb{Z}_+$. Powers of $\Delta$ are understood in the sense of composition, i.e. we have $\Delta^k b = \Delta(\Delta^{k-1}b)$ for $k \in \{2, 3, \ldots\}$. Sometimes we use the following version of Abel's summation formula.

**Lemma 1** *For $n \in \mathbb{Z}_+$, let $a_n, b_n \in \mathbb{R}$, $A_n = \sum_{i=0}^n a_i$. If $\sum_{n=0}^\infty |a_n| < \infty$ and $\sum_{n=0}^\infty |b_n| < \infty$ then*

$$\sum_{n=0}^\infty a_n\, b_n = \sum_{n=0}^\infty a_n \sum_{m=n}^\infty \Delta b_m = \sum_{m=0}^\infty A_m\, \Delta b_m.$$

For the proof of Theorem 1, we need the following lemma.

**Lemma 2** *Let the assumptions of Theorem 1 be valid and set, for $y \in \mathbb{R}$, $b_n(y) = F_{S_n}(y)$ and, for $n \in \mathbb{Z}_+$, $A_n = F_M(n) - F_N(n)$.*

(a) *For $y \in \mathbb{R}$, we have*

$$F_{S_M}(y) - F_{S_N}(y) = \sum_{n=0}^{\infty} A_n\, \Delta b_n(y).$$

(b) *If* $\mathrm{E}(X_1) < \infty$, *then, for* $t \in \mathbb{R}$,

$$\pi_{S_M}(t) - \pi_{S_N}(t) = \sum_{n=1}^{\infty} (\pi_M(n) - \pi_N(n)) \int_0^t \Delta^2 b_{n-1}(y)\, \mathrm{d}y.$$

**Proof.** We may assume that, for all $i$, $X_i \neq 0$ with positive probability. Assertion (a) follows with the help of Abel's summation formula. Here, it has been used that $\sum_{n=0}^{\infty} F_{S_n}(y)$ is a renewal function, which is bounded on intervals of finite lengths (for example, see Feller 1971, p. 359). We now prove (b). We may assume that $t \in [0, \infty)$. Then

$$\pi_{S_M}(t) - \pi_{S_N}(t) = \int_t^{\infty} (F_{S_N}(y) - F_{S_M}(y))\, \mathrm{d}y = \int_0^t (F_{S_M}(y) - F_{S_N}(y))\, \mathrm{d}y,$$

where we used that, under the present assumptions, $\mathrm{E}(S_M) = \mathrm{E}(S_N) < \infty$. Application of Abel's summation formula to assertion (a) gives

$$\begin{aligned} F_{S_M}(y) - F_{S_N}(y) &= \sum_{n=0}^{\infty} \Big(\sum_{m=0}^{n} A_m\Big) \Delta^2 b_n(y) \\ &= \sum_{n=1}^{\infty} [\pi_M(n) - \pi_N(n)] \Delta^2 b_{n-1}(y) \end{aligned}$$

for $y \in \mathbb{R}$, where we took into account that $\sum_{n=0}^{\infty} |A_n| \leq \mathrm{E}(M+N) < \infty$ and that, for $k \in \mathbb{Z}_+$

$$\sum_{n=0}^{k-1} A_n = -\sum_{n=k}^{\infty} A_n = \pi_M(k) - \pi_N(k).$$

To complete the proof of (b), we use Fubini's theorem, which can be applied, since

$$\int_0^t \sum_{n=1}^{\infty} |(\pi_M(n) - \pi_N(n)) \Delta^2 b_{n-1}(y)|\, \mathrm{d}y \leq 4\mathrm{E}(M+N) \int_0^t \sum_{n=0}^{\infty} F_{S_n}(y)\, \mathrm{d}y < \infty. \quad \square$$

**Proof of Theorem 1.** Let $b_n(y)$ and $A_n$ be defined as in Lemma 2. Using Lemma 2(a), we obtain, for all $c, x \in \mathbb{R}$,

$$\begin{aligned} P(S_M + Y \leq x) - P(S_N + Y \leq x) &= \mathrm{E}[F_{S_M}(x - Y) - F_{S_N}(x - Y)] \\ &= \mathrm{E}\Big[\sum_{n=0}^{\infty} A_n\, \Delta b_n(x - Y)\Big] \\ &= \sum_{n=0}^{\infty} A_n (\mathrm{E}[\Delta b_n(x - Y)] - c), \end{aligned}$$

where the latter equality follows from Fubini's theorem and $\mathrm{E}(M) = \mathrm{E}(N) < \infty$, i.e. $\sum_{n=0}^{\infty} A_n = 0$. It follows that

$$d_{\mathrm{KM}}(S_M + Y,\, S_N + Y) \leq d_{\mathrm{FM}}(M,\, N) \sup_{n\in\mathbb{Z}_+} \sup_{x\in\mathbb{R}} |\mathrm{E}[\Delta b_n(x - Y)] - c|.$$

Since the $X_i$ are non-negative, we have, for all $n \in \mathbb{Z}_+$ and $x \in \mathbb{R}$, $\Delta b_n(x-Y) \geq 0$, giving

$$\begin{aligned} 0 &\leq \mathrm{E}[\Delta b_n(x - Y)] \\ &= P(S_n + Y \leq x) - P(S_{n+1} + Y \leq x) \\ &= \mathrm{E}[F_Y(x - S_n) - F_{X_1+Y}(x - S_n)] \\ &\leq d_{\mathrm{KM}}(Y,\, X_1 + Y). \end{aligned}$$

Hence, if we set $c = 2^{-1} d_{\mathrm{KM}}(Y,\, X_1 + Y)$, we obtain

$$|\mathrm{E}[\Delta b_n(x - Y)] - c| \leq \frac{1}{2}\, d_{\mathrm{KM}}(Y,\, X_1 + Y).$$

Assertion (a) immediately follows. Now we prove (b). For $t \in \mathbb{R}$, we have

$$\begin{aligned} \pi_{S_M+Y}(t) - \pi_{S_N+Y}(t) &= \mathrm{E}[(S_M + Y - t)_+ - (S_N + Y - t)_+] \\ &= \mathrm{E}[\pi_{S_M}(t - Y) - \pi_{S_N}(t - Y)] \\ &= \mathrm{E}\Big[\sum_{n=1}^{\infty} (\pi_M(n) - \pi_N(n)) \int_0^{t-Y} \Delta^2 b_{n-1}(y)\,\mathrm{d}y\Big], \end{aligned}$$

where we used Lemma 2(b). Since the $X_n$ are non-negative, independent and identically distributed with finite mean, $\pi_{S_n}(y)$ is a convex sequence in $n \in \mathbb{Z}_+$ for $y \in \mathbb{R}$ being fixed, i.e. $\Delta^2 \pi_{S_n}(y) \geq 0$ for all $n \in \mathbb{Z}_+$ (e.g. see Müller and Stoyan (2002, p. 160)). In fact, this follows from the equalities

$$\begin{aligned} \Delta^2 \pi_{S_n}(y) &= \pi_{S_{n+2}}(y) - 2\pi_{S_{n+1}}(y) + \pi_{S_n}(y) \\ &= \mathrm{E}[(S_n + X_{n+1} + X_{n+2} - y)_+ - (S_n + X_{n+1} - y)_+ \\ &\qquad - (S_n + X_{n+2} - y)_+ + (S_n - y)_+], \end{aligned}$$

and the obvious fact that, for all $\alpha, \beta, \gamma \geq 0$,

$$(\alpha + \beta - y)_+ + (\alpha + \gamma - y)_+ \leq (\alpha + \beta + \gamma - y)_+ + (\alpha - y)_+.$$

From this, we see that, for $n \in \mathbb{N}$, $\int_0^{t-Y} \Delta^2 b_{n-1}(y)\,\mathrm{d}y \geq 0$, and therefore we arrive at

$$d_{\mathrm{SL}}(S_M + Y,\, S_N + Y) \leq \tilde{d}_{\mathrm{SL}}(M,\, N) \sup_{n\in\mathbb{N}} \sup_{t\in\mathbb{R}} \mathrm{E}\Big[\int_0^{t-Y} \Delta^2 b_{n-1}(y)\,\mathrm{d}y\Big].$$

For all $n \in \mathbb{N}$ and $t \in \mathbb{R}$, we have, by conditioning on the values of $S_{n-1}$, $X_n$, and $X_{n+1}$,

$$\begin{aligned}
\mathrm{E}\Big[\int_0^{t-Y} \Delta^2 b_{n-1}(y)\,\mathrm{d}y\Big] &= \mathrm{E}\Big[\int_0^{t-Y} [P(S_{n-1} \leq y) - P(S_n \leq y) \\
&\qquad - P(S_{n-1} + X_{n+1} \leq y) + P(S_{n+1} \leq y)]\,\mathrm{d}y\Big] \\
&= \int \mathrm{E}[Z_{a,b,c}(t-Y)]\,\mathrm{d}\mathcal{L}((S_{n-1}, X_n, X_{n+1}))(a,b,c),
\end{aligned}$$

where, for $x \in \mathbb{R}$,

$$\begin{aligned}
Z_{a,b,c}(x) &= \int_0^x (\mathbf{1}_{(-\infty,\,y]}(a) - \mathbf{1}_{(-\infty,\,y]}(a+b) \\
&\qquad - \mathbf{1}_{(-\infty,\,y]}(a+c) + \mathbf{1}_{(-\infty,\,y]}(a+b+c))\,\mathrm{d}y \\
&= \begin{cases} x - a, & \text{if} \quad a < x \leq a + (b \wedge c), \\ b \wedge c, & \text{if} \quad a + (b \wedge c) < x \leq a + (b \vee c), \\ a + b + c - x, & \text{if} \quad a + (b \vee c) < x \leq a + b + c, \\ 0, & \text{otherwise} \end{cases} \\
&\leq (b \wedge c)\,\mathbf{1}_{(a,\,a+b+c]}(x).
\end{aligned}$$

Here, for a set $A$, $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ otherwise. This leads to

$$\begin{aligned}
&\mathrm{E}\Big[\int_0^{t-Y} \Delta^2 b_{n-1}(y)\,\mathrm{d}y\Big] \\
&\leq \int (b \wedge c)\,P(t - Y \in (a,\, a+b+c])\,\mathrm{d}\mathcal{L}((S_{n-1},\, X_n,\, X_{n+1}))(a,b,c) \\
&\leq \mathrm{E}[(X_n \wedge X_{n+1})\,\mathrm{Conc}^-(\mathcal{L}(Y)\,;\, X_n + X_{n+1})].
\end{aligned}$$

Assertion (b) follows from the inequalities above. □

The proof of Proposition 1 requires the following three lemmas.

**Lemma 3** *Let the assumptions of Proposition 1 be valid. Further, let $Y_1, \dots, Y_n$ be random variables with distributions $\mathcal{L}(Y_j) = \mathrm{CPo}(p_j,\, Q_j)$ for $j \in \{1, \dots, n\}$. We assume that all $X_1, \dots, X_n, Y_1, \dots, Y_n$ are independent. For $i \in \{1, \dots, n\}$ being fixed, set $Z_i = \sum_{j=1}^{i} X_j + \sum_{j=i+1}^{n} Y_j$ and $Z_i' = Z_i - X_i$. Then, for all $t \in [0, \infty)$,*

$$\mathrm{Conc}(\mathcal{L}(Z_i')\,;\, t) \leq \frac{1}{1-p_i}\mathrm{Conc}(\mathcal{L}(Z_i)\,;\, t), \qquad \mathrm{Conc}(\mathcal{L}(Z_i)\,;\, t) \leq \frac{\pi^2}{4}\mathrm{Conc}(\tilde{H}\,;\, t).$$

*The above inequalities remain valid, if* Conc *is everywhere replaced with* $\mathrm{Conc}^-$.

**Proof.** For the proof with respect to $\mathrm{Conc}^-$, see Hipp and Michel (1990, pp. 52–53) or, for a preliminary version, Hipp (1985, p. 231), where the main argument is a suitable smoothing lemma for arbitrary probability measures. The proof is completed by using the continuity properties of concentration functions (see Lemma 9 below). □

**Lemma 4** (a) *Let $X$ and $Y$ be two real-valued random variables. If $\mu = \mathrm{E}|X| < \infty$, then*

$$d_{\mathrm{KM}}(Y,\, X+Y) \leq c \,\mathrm{Conc}^-(\mathcal{L}(Y)\,;\,\mu),$$

*where $c = 2$. If $X$ is almost surely constant, then we can set $c = 1$.*

(b) *Let $Y$ be a real valued random variable and let $X_1$, $X_2$ be non-negative, independent and identically distributed random variables with $\mathrm{E}(X_1^2) < \infty$. Then*

$$\begin{aligned}&\mathrm{E}[(X_1 \wedge X_2)\,\mathrm{Conc}^-(\mathcal{L}(Y)\,;\,X_1+X_2)]\\ &\qquad\leq\; 2\Big(\mu + c'\Big(\mu + \frac{\mu^{(2)}}{\mu}\Big)\Big)\mathrm{Conc}^-(\mathcal{L}(Y)\,;\,\mu),\end{aligned}$$

*where $\mu = \mathrm{E}(X_1)$, $\mathrm{E}(X_1^2) = \mu^{(2)}$ and $c' = 1/4$. If $X_1$ is almost surely constant, then we can set $c' = 0$.*

**Proof.** Assertion (a) was implicitly shown in Hipp and Michel (1990, p. 52); see also Hipp (1985, pp. 230–231). In fact, the argument is the following: we may assume that $\mu > 0$. For $y \in \mathbb{R}$, we have

$$|P(Y \leq y) - P(X+Y \leq y)| \leq \int_{\mathbb{R}} |P(Y \leq y) - P(Y \leq y-x)|\,\mathrm{d}\mathcal{L}(X)(x).$$

The integrand is equal to $P(Y \in I(x,y))$, where $I(x,y) = (y \wedge (y-x),\, y \vee (y-x)]$ is a half-open interval with length $|x|$. Dividing this interval into smaller ones, we see that

$$P(Y \in I(x,y)) \leq \left\lceil \frac{|x|}{\mu} \right\rceil \mathrm{Conc}^-(\mathcal{L}(Y)\,;\,\mu),$$

where, for $x \in \mathbb{R}$, $\lceil x \rceil \in \mathbb{Z}$ is defined by $x \leq \lceil x \rceil < x+1$. Therefore

$$d_{\mathrm{KM}}(Y,\, X+Y) \leq \mathrm{Conc}^-(\mathcal{L}(Y)\,;\,\mu)\,\mathrm{E}\left\lceil \frac{|X|}{\mu} \right\rceil,$$

from which (a) follows. Assertion (b) can be shown in the same way. Indeed, we have

$$\mathrm{E}[(X_1 \wedge X_2)\,\mathrm{Conc}^-(\mathcal{L}(Y)\,;\,X_1+X_2)]$$

$$
\begin{aligned}
&\leq \mathrm{E}\Big[(X_1 \wedge X_2)\Big(1+\frac{X_1+X_2}{\mu}\Big)\Big]\mathrm{Conc}^-(\mathcal{L}(Y)\,;\,\mu)\\
&= \Big(\mathrm{E}(X_1\wedge X_2)+\frac{1}{\mu}\,\mathrm{E}(X_1\wedge X_2)^2+\mu\Big)\mathrm{Conc}^-(\mathcal{L}(Y)\,;\,\mu)
\end{aligned}
$$

and $X_1 \wedge X_2 \leq (X_1+X_2)/2$. □

Let $\mathrm{Bi}(n,p)$ denote the binomial distribution with parameters $n \in \mathbb{N}$ and $p \in [0,1]$.

**Lemma 5** *For* $p \in [0,1]$,

$$
d_{\mathrm{FM}}(\mathrm{Bi}(1,p),\,\mathrm{Po}(p)) = 2(\mathrm{e}^{-p}-1+p) \leq p^2, \qquad \tilde{d}_{\mathrm{SL}}(\mathrm{Bi}(1,p),\,\mathrm{Po}(p)) = \frac{p^2}{2}.
$$

**Proof.** The assertions are easily shown. Also, see Roos (2001, Proposition 1 and Remark after Proposition 2). □

**Proof of Proposition 1.** For $i \in \{1,\ldots,n\}$, let

$$
P_i = \mathcal{L}(X_i), \quad \hat{P}_i = \mathrm{CPo}(p_i,\,Q_i), \quad M_i' = \Big(\mathop{\ast}_{j=1}^{i-1} P_j\Big) \ast \Big(\mathop{\ast}_{j=i+1}^{n} \hat{P}_j\Big).
$$

Then, according to Corollary 1 and Lemmas 4 and 5,

$$
\begin{aligned}
d_{\mathrm{KM}}(\mathcal{L}(S_n),\,H) &\leq \frac{1}{2}\sum_{i=1}^{n} d_{\mathrm{FM}}(\mathrm{Bi}(1,\,p_i),\,\mathrm{Po}(p_i))\, d_{\mathrm{KM}}(M_i',\,Q_i \ast M_i')\\
&\leq \frac{1}{2}\sum_{i=1}^{n} c_i\, p_i^2\,\mathrm{Conc}^-(M_i'\,;\,\mu_i)
\end{aligned}
$$

and similarly

$$
d_{\mathrm{SL}}(\mathcal{L}(S_n),\,H) \leq \sum_{i=1}^{n} p_i^2\Big(\mu_i + c_i'\Big(\mu_i+\frac{\mu_i^{(2)}}{\mu_i}\Big)\Big)\mathrm{Conc}^-(M_i'\,;\,\mu_i).
$$

Lemma 3 gives

$$
\mathrm{Conc}^-(M_i'\,;\,\mu_i) \leq \frac{\pi^2}{4(1-p_i)}\,\mathrm{Conc}^-(\tilde{H}\,;\,\mu_i),
$$

which completes the proof. □

**Proof of Proposition 2.** For $i \in \{1,\ldots,n\}$, let $P_i = \mathcal{L}(X_i)$,

$$
\begin{aligned}
R &= \mathcal{L}(S_n), \qquad R_K^{(i)} = \sum_{k=1}^{K}\frac{(-1)^{k+1}}{k}p_i^k(Q_i-\varepsilon_0)^{\ast k},\\
R^{(i)} &= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}p_i^k(Q_i-\varepsilon_0)^{\ast k}, \qquad U^{(i)} = R_K^{(i)} - R^{(i)},\\
H_K^{(i)} &= \exp(R_K^{(i)}), \qquad M_i'' = \Big(\mathop{\ast}_{j=1}^{i-1} P_j\Big) \ast \Big(\mathop{\ast}_{j=i+1}^{n} H_K^{(j)}\Big).
\end{aligned}
$$

Then the telescopic sum decomposition (cf. proof of Corollary 1) gives

$$d_{\mathrm{KM}}(R,\, H_K) \leq \sum_{i=1}^{n} \sup_{x\in\mathbb{R}} \Big|[M_i'' * (P_i - H_K^{(i)})]((-\infty, x])\Big| =: T.$$

Using the telescopic sum decomposition again, we obtain, for $i \in \{1,\ldots,n\}$,

$$R - P_i * M_i'' = \Big[\mathop{\ast}_{j=i+1}^{n} P_j - \mathop{\ast}_{j=i+1}^{n} H_K^{(j)}\Big] * \Big(\mathop{\ast}_{j=1}^{i} P_j\Big) = \sum_{j=i+1}^{n} M_j'' * (P_j - H_K^{(j)}).$$

Since $P_i = \exp(R^{(i)})$, this yields

$$\begin{aligned} M_i'' * (P_i - H_K^{(i)}) &= M_i'' * P_i * (\varepsilon_0 - \exp(U^{(i)})) \\ &= (R + M_i'' * P_i - R) * (\varepsilon_0 - \exp(U^{(i)})) \\ &= \Big(R - \sum_{j=i+1}^{n} M_j'' * (P_j - H_K^{(j)})\Big) * (\varepsilon_0 - \exp(U^{(i)})). \end{aligned}$$

In view of Abel's summation formula, we see that the second convolution factor is equal to

$$\varepsilon_0 - \exp(U^{(i)}) = \sum_{r=0}^{\infty} a_r^{(i)} Q_i^{*r} = \sum_{r=0}^{\infty} A_r^{(i)} Q_i^{*r} * (\varepsilon_0 - Q_i), \tag{13}$$

where the coefficients $a_r^{(i)}$ are real-valued and $A_r^{(i)} = \sum_{m=0}^{r} a_m^{(i)}$ for $r \in \mathbb{Z}_+$. In fact, (13) is valid, since it can be shown that $B^{(i)} := \sum_{r=0}^{\infty} |A_r^{(i)}| < \infty$ (see below). Hence, for $i \in \{1,\ldots,n\}$,

$$\sup_{x\in\mathbb{R}} |[M_i'' * (P_i - H_K^{(i)})]((-\infty,\, x])| \leq B^{(i)}(d_{\mathrm{KM}}(R,\, Q_i * R) + 2T).$$

This implies

$$T \leq \sum_{i=1}^{n} B^{(i)}(d_{\mathrm{KM}}(R,\, Q_i * R) + 2T),$$

and therefore, letting $B = \sum_{i=1}^{n} B^{(i)}$,

$$(1 - 2B)T \leq \sum_{i=1}^{n} B^{(i)} d_{\mathrm{KM}}(R,\, Q_i * R).$$

Here, it has been used that, since $M_i'' * (P_i - H_K^{(i)})$ is a finite signed measure for all $i$, $T$ must be finite. In order to give an estimate for $B$, we use, for a power series $g(z) = \sum_{i=0}^{\infty} g_i z^i$ with $|z| \leq 1$, the notation $\|g(z)\| = \sum_{i=0}^{\infty} |g_i|$. Further, we make use of the simple property that $\|g(z)\,\tilde{g}(z)\| \leq \|g(z)\|\,\|\tilde{g}(z)\|$, where $g(z)$ and $\tilde{g}(z)$ are two such power series. For $|z| \leq 1$, let

$$G_K^{(i)}(z) = -\sum_{k=K+1}^{\infty} \frac{(-1)^{k+1}}{k} p_i^k (z-1)^k.$$

Then it follows that, for all $i \in \{1, \dots, n\}$,

$$\begin{aligned} B^{(i)} &= \Big\| \frac{1}{1-z}(1 - \exp(G_K^{(i)}(z))) \Big\| \\ &\leq \Big\| \frac{G_K^{(i)}(z)}{z-1} \Big\| \sum_{m=1}^{\infty} \frac{\|G_K^{(i)}(z)\|^{m-1}}{m!} \\ &\leq \frac{1}{2}\Big( \exp\Big( \sum_{k=K+1}^{\infty} \frac{(2p_i)^k}{k} \Big) - 1 \Big) \\ &\leq \frac{1}{2}(\mathrm{e}^{\tau(i,K)} - 1), \end{aligned}$$

giving $B \leq \delta/2$. Since $d_{\mathrm{KM}}(R,\, Q_i * R) \leq 4^{-1}\pi^2 c_i \mathrm{Conc}^-(\tilde{H}\,;\, \mu_i)$ (cf. Lemmas 4(a) and 3), we arrive at the first inequality. The second assertion is shown in the same manner. Here, we may assume that, for all $i$, $\mu_i < \infty$ and $\mu_i^{(2)} < \infty$. Now

$$d_{\mathrm{SL}}(R,\, H_K) \leq \sum_{i=1}^{n} \sup_{x \in \mathbb{R}} \Big| \int_x^{\infty} [M_i'' * (P_i - H_K^{(i)})]((y,\, \infty))\,\mathrm{d}y \Big| =: \tilde{T}.$$

By using Abel's summation formula, we have, for $i \in \{1, \dots, n\}$,

$$\varepsilon_0 - \exp(U^{(i)}) = \sum_{r=0}^{\infty} A_r^{(i)} Q_i^{*r} * (\varepsilon_0 - Q_i) = \sum_{r=0}^{\infty} \tilde{A}_r^{(i)} Q_i^{*r} * (\varepsilon_0 - Q_i)^{*2},$$

with $\tilde{A}_r^{(i)} = \sum_{m=0}^{r} A_m^{(i)}$ for $r \in \mathbb{Z}_+$. This leads to

$$M_i'' * (P_i - H_K^{(i)}) = \Big( R - \sum_{j=i+1}^{n} M_j'' * (P_j - H_K^{(j)}) \Big) * \sum_{r=0}^{\infty} \tilde{A}_r^{(i)} Q_i^{*r} * (\varepsilon_0 - Q_i)^{*2}.$$

Hence

$$\begin{aligned} &\sup_{x \in \mathbb{R}} \Big| \int_x^{\infty} [M_i'' * (P_i - H_K^{(i)})]((y,\, \infty))\,\mathrm{d}y \Big| \\ &\qquad \leq \sum_{r=0}^{\infty} |\tilde{A}_r^{(i)}| \Big( \sup_{x \in \mathbb{R}} \Big| \int_x^{\infty} [Q_i^{*r} * (\varepsilon_0 - Q_i)^{*2} * R]((y,\, \infty))\,\mathrm{d}y \Big| + 4\tilde{T} \Big). \end{aligned}$$

It is easy to see that, for $r \in \mathbb{Z}_+$,

$$\sup_{x \in \mathbb{R}} \Big| \int_x^{\infty} [Q_i^{*r} * (\varepsilon_0 - Q_i)^{*2} * R]((y,\, \infty))\,\mathrm{d}y \Big| = \sup_{x \in \mathbb{R}} \Big| \mathrm{E} \int_0^{x - S_n} \Delta^2 b_r^{(i)}(y)\,\mathrm{d}y \Big|,$$

where $b_r^{(i)}(y) = Q_i^{*r}((-\infty,\, y])$. Proceeding as in the proof of Theorem 1(b), we obtain together with Lemma 4(b) that

$$\tilde{T} \leq \sum_{i=1}^{n} \sum_{r=0}^{\infty} |\tilde{A}_r^{(i)}| \Big(2\Big(\mu_i + c_i'\Big(\mu_i + \frac{\mu_i^{(2)}}{\mu_i}\Big)\Big)\mathrm{Conc}^-(R\,;\,\mu_i) + 4\tilde{T}\Big),$$

giving

$$(1 - 4\tilde{B})\tilde{T} \leq 2\sum_{i=1}^{n} \tilde{B}^{(i)}\Big(\mu_i + c_i'\Big(\mu_i + \frac{\mu_i^{(2)}}{\mu_i}\Big)\Big)\mathrm{Conc}^-(R\,;\,\mu_i),$$

where $\tilde{B} = \sum_{i=1}^n \tilde{B}^{(i)}$ and $\tilde{B}^{(i)} = \sum_{r=0}^\infty |\tilde{A}_r^{(i)}|$. Here, it has been used that $\tilde{T} < \infty$, which can easily be shown by using the simple inequality

$$\sup_{t\in\mathbb{R}} |\pi_{Q_1' * Q_2'}(t)| \leq \sup_{t\in\mathbb{R}} |\pi_{Q_1'}(t)|\, |Q_2'|(\mathbb{R})$$

for two finite signed measures $Q_1'$ and $Q_2'$ on $\mathbb{R}$. Similar to the above,

$$\tilde{B}^{(i)} = \Big\| \frac{1}{(1-z)^2}(1 - \exp(G_K^{(i)}(z)))\Big\| \leq \frac{1}{4}(\mathrm{e}^{\tau(i,K)} - 1),$$

and hence, we obtain $\tilde{B} \leq \delta/4$. The second assertion now follows. □

For the proofs of Proposition 3, Lemma 6 below, and Proposition 5, we use a splitting technique by Lévy (cf. Le Cam 1986, p. 412).

**Proof of Proposition 3.** Let $s \in [0,\infty)$. The proof is based on the decomposition

$$Q = c_1\, Q_1 + c_2\, Q_2 + c_3\, Q_3,$$

where $Q_1$, $Q_2$, and $Q_3$ are probability measures concentrated on $(-\infty,\, -s)$, $[-s,\, s]$, and on $(s,\, \infty)$, respectively, and

$$c_1 = Q((-\infty,\, -s)), \qquad c_2 = Q([-s,\, s]), \qquad c_3 = Q((s,\, \infty)).$$

Then, for $t \in (0,\infty)$, $\mathrm{CPo}(t,\, Q) = *_{i=1}^3 \mathrm{CPo}(tc_i,\, Q_i)$ and therefore, by Lemma 8(c) below,

$$\mathrm{Conc}(\mathrm{CPo}(t,\, Q)\,;\, s) \leq \min\{\mathrm{Conc}(\mathrm{CPo}(tc_1,\, Q_1)\,;\, s),\, \mathrm{Conc}(\mathrm{CPo}(tc_3,\, Q_3)\,;\, s)\}.$$

We obtain

$$\begin{aligned}\mathrm{Conc}(\mathrm{CPo}(tc_1,\, Q_1)\,;\, s) &= \sup_{x\in\mathbb{R}} \Big(\sum_{n=0}^{\infty} \mathrm{po}(n,\, tc_1)\, Q_1^{*n}([x,\, x+s])\Big) \\ &\leq \Big(\sup_{n\in\mathbb{Z}_+} \mathrm{po}(n,\, tc_1)\Big) \sup_{x\in\mathbb{R}} \sum_{n=0}^{\infty} Q_1^{*n}([x,\, x+s]).\end{aligned}$$

It is well-known that, for $y \in (0, \infty)$,

$$\sup_{n\in\mathbb{Z}_+} \mathrm{po}(n,\, y) \leq \frac{1}{\sqrt{2\mathrm{e}\, y}}; \tag{14}$$

see, for example, Barbour, Holst, and Janson (1992, p. 262) or Hipp and Michel (1990, pp. 46–47). Further, for all $x \in \mathbb{R}$, it can be shown that $\sum_{n=0}^{\infty} Q_1^{*n}([x,\, x+s]) \leq 1$. Indeed, if $T_1, T_2, \ldots$ are independent and identically distributed random variables with $\mathcal{L}(T_1) = Q_1$, then we may assume that, for all $i \in \mathbb{N}$, $T_i < -s$, and therefore

$$\sum_{n=0}^{\infty} Q_1^{*n}([x,\, x+s]) = P\Big(\bigcup_{n=0}^{\infty}\Big\{\sum_{i=1}^{n} T_i \in [x,\, x+s]\Big\}\Big) \leq 1.$$

Hence

$$\mathrm{Conc}(\mathrm{CPo}(tc_1,\, Q_1)\,;\, s) \leq \frac{1}{\sqrt{2\mathrm{e}\, tc_1}}.$$

Similarly, $\mathrm{Conc}(\mathrm{CPo}(tc_3,\, Q_3)\,;\, s) \leq (2\mathrm{e}\, tc_3)^{-1/2}$. Combining the estimates above, (5) is shown. Inequality (6) can be derived from (5) by using Lemma 9 below. Since $\mathrm{CPo}(t, \varepsilon_1) = \mathrm{Po}(t)$ for $t \in [0, \infty)$ and since, in (14), equality holds for $y = 1/2$, we see that the remaining part of the assertion is true. □

For the proof of Proposition 5, we need the following lemma, which is similar to Proposition 2 in Le Cam (1986, pp. 409–410). However, there are some differences. In contrast to Lemma 6 below, in Le Cam's Proposition 2, it was assumed that the summands $X_i$ have not necessarily identical but symmetric distributions.

**Lemma 6** *Let $n \in \mathbb{N}$ and $X_1, \ldots, X_n$ be independent and identically distributed random variables. Set $S_n = \sum_{i=1}^n X_i$. Let $x \in \mathbb{R}$, $t > 0$ be fixed. We assume that the $X_i$ admit the decomposition $\mathcal{L}(X_i) = \mathcal{L}(I_iY_i + (1 - I_i)Z_i)$ for $i \in \{1, \ldots, n\}$, where $\{I_i\}$, $\{Y_i\}$, and $\{Z_i\}$ are sets of identically distributed random variables with $\mathcal{L}(I_i) = \mathrm{Bi}(1, 1/2)$, $P(Y_i \leq x) = P(Z_i \geq x + t) = 1$. We assume that all $I_i$, $Y_i$, $Z_i$ are independent. Then*

$$\mathrm{Conc}^-(\mathcal{L}(S_n)\,;\, t) \leq \frac{1}{\sqrt{n+1}}.$$

**Proof.** Set $T_n = \sum_{i=1}^n I_i$ and, for $m \in \{0, \ldots, n\}$, $\tilde{Z}_m = \sum_{i=1}^m Y_i + \sum_{i=1}^{n-m} Z_i$. For $y \in \mathbb{R}$, we then have

$$P(S_n \in (y,\, y+t)) \;=\; \sum_{m=0}^{n} P(T_n = m)\, P(\tilde{Z}_m \in (y,\, y+t))$$

$$\begin{aligned}
&\le \Big(\sup_{m\in\mathbb{Z}_+} P(T_n=m)\Big)\sum_{m=0}^{n} P(\tilde{Z}_m\in(y,\,y+t))\\
&\le \frac{1}{\sqrt{n+1}},
\end{aligned}$$

where we used that $\sup_{m\in\mathbb{Z}_+} P(T_n=m)\le(1+n)^{-1/2}$ (for example, see Le Cam 1986, proof of Proposition 2, p. 410) and that, since $\tilde{Z}_m-\tilde{Z}_{m+1}\ge t$ almost surely for $m\in\{0,\dots,n-1\}$, the events $\{\tilde{Z}_m\in(y,\,y+t)\}$ for $m\in\{0,\dots,n\}$ are pairwise disjoint. □

Proposition 5 can be proved by adapting the proof of Theorem 2 in Le Cam (1986, p. 411); cf. Remark 8 of the present paper. In what follows, we give an alternative direct proof.

**Proof of Proposition 5.** Let $t\in(0,\infty)$ and let $x=x_t\in\mathbb{R}$ be a median of the distribution function $2^{-1}(F(y)+F(y+t))$ for $y\in\mathbb{R}$, where $F$ is the distribution function of $X_1$. This means that

$$F(x-)+F((x+t)-)\le 1\le F(x)+F(x+t),$$

where $F(x-)=\lim_{y\uparrow x}F(y)$. Therefore $a\in[0,1]$ exists such that

$$q:=F(x-)+a\,P(X_1=x)=1-F((x+t)-)-a\,P(X_1=x+t).$$

This leads to

$$1-P(X_1\in[x,\,x+t])\le 2q\le 1-P(X_1\in(x,\,x+t)). \tag{15}$$

In particular, $q\le 1/2$. Let us assume that $q>0$. For $y\in\mathbb{R}$, set

$$\begin{aligned}
F_1(y) &= \frac{F(y)}{q}\mathbf{1}_{(-\infty,\,x)}(y)+\mathbf{1}_{[x,\,\infty)}(y),\\
F_2(y) &= \frac{F(y)-(1-q)}{q}\mathbf{1}_{[x+t,\,\infty)}(y),\\
F_3(y) &= \frac{1}{2}(F_1(y)+F_2(y)),\\
F_4(y) &= \begin{cases} \frac{F(y)-q}{1-2q}\mathbf{1}_{[x,\,x+t)}(y)+\mathbf{1}_{[x+t,\,\infty)}(y), & \text{if } q<\frac{1}{2},\\ \mathbf{1}_{[x,\,\infty)}(y), & \text{if } q=\frac{1}{2}.\end{cases}
\end{aligned}$$

It is easy to verify that the $F_1,\dots,F_4$ are distribution functions with

$$F=2qF_3+(1-2q)F_4.$$

Further, the distributions with the distribution functions $F_1,F_2,F_4$ are concentrated on $(-\infty,\,x]$, $[x+t,\,\infty)$, and $[x,\,x+t]$, respectively. Let $\{Y_1,\dots,Y_n\}$,

$\{Z_1,\ldots,Z_n\}$ and $\{I_1,\ldots,I_n\}$ be families of identically distributed random variables with $F_{Y_i}=F_3$, $F_{Z_i}=F_4$ and $\mathcal{L}(I_i)=\mathrm{Bi}(1,\,2q)$ for $i\in\{1,\ldots,n\}$, where we assume that all $Y_i$, $Z_i$, $I_i$ for $i\in\{1,\ldots,n\}$ are independent. Then $S_n$ is equal in distribution to

$$\sum_{i=1}^{n}[I_iY_i+(1-I_i)Z_i].$$

Set $T_n=\sum_{i=1}^n I_i$ and, for $m\in\{0,\ldots,n\}$, $R_m=\sum_{i=1}^m Y_i$. For $y\in\mathbb{R}$, we now obtain

$$\begin{aligned}P(S_n\in(y,\,y+t)) &= \sum_{m=0}^{n}P(T_n=m)\,P\Big(R_m+\sum_{i=1}^{n-m}Z_i\in(y,\,y+t)\Big)\\ &\le \sum_{m=0}^{n}P(T_n=m)\,\mathrm{Conc}^-(\mathcal{L}(R_m)\,;\,t),\end{aligned}$$

where we used Lemma 8 below. From Lemma 6, we see that $\mathrm{Conc}^-(\mathcal{L}(R_m)\,;\,t)\le(m+1)^{-1/2}$. Therefore, using Jensen's inequality, the equality

$$P(T_n+1=m)=\frac{m}{(n+1)2q}\,P(T_{n+1}=m)$$

for $m\in\{0,\ldots,n+1\}$ and (15), we derive

$$\begin{aligned}\mathrm{Conc}^-(\mathcal{L}(S_n)\,;\,t) &\le \mathrm{E}\frac{1}{\sqrt{T_n+1}}\le\Big(\mathrm{E}\frac{1}{T_n+1}\Big)^{1/2}=\Big(\frac{1-(1-2q)^{n+1}}{(n+1)\,2q}\Big)^{1/2}\\ &\le \Big(\frac{1-[\mathrm{Conc}(\mathcal{L}(X_1)\,;\,t)]^{n+1}}{(n+1)(1-\mathrm{Conc}(\mathcal{L}(X_1)\,;\,t))}\Big)^{1/2},\end{aligned}$$

where $t\in(0,\,\infty)$. In the case $q=0$, the upper bound we have just proved can be set to be one by a continuity argument, since here we have $P(X_1\in[x,\,x+t])=1$ and therefore $\mathrm{Conc}(\mathcal{L}(X_1)\,;\,t)=1$. The proof is completed by using Lemma 9 below. $\square$

For the proof of Proposition 4, we need the following lemma.

**Lemma 7** *Let the assumptions of Proposition 4 be valid. For $s,t\in(0,\infty)$ and $\alpha=1-\mathrm{Conc}(\mathcal{L}(X_1)\,;\,s)$, we have*

$$\mathrm{Conc}^-(\mathcal{L}(S_n)\,;\,t)\le\mathrm{Conc}^-(\mathcal{L}(X_1)\,;\,t)\sum_{m=0}^{n-1}\mathrm{Conc}^-(\mathcal{L}(S_m)\,;\,s+t)\alpha^{n-1-m}.$$

**Proof.** Let $x\in\mathbb{R}$ be arbitrary and set $I=(x,\,x+t]$. According to Lemma 9(d) below, $y\in\mathbb{R}$ exists such that $\alpha=P(X_1\notin J)$, where we define $J=[y,\,y+s]$.

Then we have (cf. Petrov 1995, p. 70)

$$\begin{aligned} P(S_n \in I) &= \int_{I-J} P(X_1 + x \in I,\, X_1 \in J)\, \mathrm{d}\mathcal{L}(S_{n-1})(x) \\ &\quad + \alpha \int_{-\infty}^{\infty} P(S_{n-1} + x \in I)\, \mathrm{d}\mathcal{L}(X_1 | X_1 \notin J)(x), \end{aligned}$$

where $I - J = \{z_1 - z_2 \,|\, z_1 \in I,\, z_2 \in J\} = (x - y - s,\, x - y + t]$. This yields

$$\begin{aligned} \mathrm{Conc}^-(\mathcal{L}(S_n)\,;\, t) &\leq \mathrm{Conc}^-(\mathcal{L}(X_1)\,;\, t)\mathrm{Conc}^-(\mathcal{L}(S_{n-1})\,;\, s+t) \\ &\quad + \alpha \mathrm{Conc}^-(\mathcal{L}(S_{n-1})\,;\, t). \end{aligned}$$

The assertion now follows by induction over $n$. □

**Proof of Proposition 4.** According to Lemma 9 below, it suffices to show the assertion for $\mathrm{Conc}^-$. Let $t \in (0, \infty)$. Let us first assume that $\mathrm{Conc}^-(\mathcal{L}(X_1)\,;\, t) \leq \beta$, where $\beta \in (0, 1)$. Then, by Lemma 9 below, $s \in [t, \infty)$ exists such that

$$\mathrm{Conc}^-(\mathcal{L}(X_1)\,;\, s) \leq \beta \leq \mathrm{Conc}(\mathcal{L}(X_1)\,;\, s).$$

In particular, we have $\alpha := 1 - \mathrm{Conc}(\mathcal{L}(X_1)\,;\, s) \leq 1 - \beta$. Using Lemma 7, Proposition 5 and the simple inequality $\mathrm{Conc}^-(\mathcal{L}(X_1)\,;\, s+t) \leq 2\beta$, we obtain

$$\mathrm{Conc}^-(\mathcal{L}(S_n)\,;\, t) \leq \mathrm{Conc}^-(\mathcal{L}(X_1)\,;\, t) \sum_{m=0}^{n-1} \frac{(1-\beta)^{n-1-m}}{\sqrt{(m+1)(1-2\beta)}}.$$

Set $\beta = 0.3322$. Then simple calculus shows that

$$\sqrt{n+1} \sum_{m=0}^{n-1} \frac{(1-\beta)^{n-1-m}}{\sqrt{(m+1)(1-2\beta)}} \leq 6.33, \tag{16}$$

giving the assertion in the present case. In fact, it is not difficult to prove that, if we denote the left-hand side of (16) by $f_n$, then $f_n \leq f_{n+1}$ for $n \leq 6$ and $f_n \geq f_{n+1}$ for $n \geq 7$. Therefore, $\sup_{n \in \mathbb{N}} f_n = f_7 = 6.329\ldots$. If $\mathrm{Conc}^-(\mathcal{L}(X_1)\,;\, t) \geq \beta$, then the assertion follows easily from Proposition 5. In fact,

$$\begin{aligned} \mathrm{Conc}^-(\mathcal{L}(S_n)\,;\, t) &\leq \frac{1}{\sqrt{(m+1)(1 - \mathrm{Conc}^-(\mathcal{L}(X_1)\,;\, t))}} \\ &\leq \frac{\mathrm{Conc}^-(\mathcal{L}(X_1)\,;\, t)}{\beta\sqrt{(m+1)(1 - \mathrm{Conc}^-(\mathcal{L}(X_1)\,;\, t))}}, \end{aligned}$$

where $1/\beta \leq 3.1$. □

**Proof of Corollary 2.** The assertion follows from

$$\begin{aligned}\mathrm{Conc}(\mathrm{CPo}(t,\,Q)\,;\,s) &\le \sum_{n=0}^{\infty}\mathrm{po}(n,\,t)\,\mathrm{Conc}(Q^{*n}\,;\,s)\\ &\le \mathrm{e}^{-t}+\sum_{n=1}^{\infty}\mathrm{po}(n,\,t)\,\frac{6.33\,\mathrm{Conc}(Q\,;\,s)}{\sqrt{(n+1)(1-\mathrm{Conc}(Q\,;\,s))}},\end{aligned}$$

the simple inequality $\sum_{n=1}^{\infty}\mathrm{po}(n,\,t)/\sqrt{n+1}\le(1-\mathrm{e}^{-t})/\sqrt{t}$ and Lemma 9 below. □

# 4 Appendix: Concentration functions

For the proof of the following lemmas, see Hengartner and Theodorescu (1973).

**Lemma 8** *(Basic properties of concentration functions). Let $t,s\in[0,\infty)$ and $X$ and $Y$ be independent real-valued random variables. Then*

(a) $\mathrm{Conc}(\mathcal{L}(X)\,;\,s)\le\mathrm{Conc}(\mathcal{L}(X)\,;\,s+t)$,

(b) $\mathrm{Conc}(\mathcal{L}(X)\,;\,s+t)\le\mathrm{Conc}(\mathcal{L}(X)\,;\,s)+\mathrm{Conc}(\mathcal{L}(X)\,;\,t)$,

(c) $\mathrm{Conc}(\mathcal{L}(X+Y)\,;\,s)\le\min\{\mathrm{Conc}(\mathcal{L}(X)\,;\,s),\,\mathrm{Conc}(\mathcal{L}(Y)\,;\,s)\}$.

(d) *The assertions* (a)–(c) *also hold if* Conc *is everywhere replaced with* $\mathrm{Conc}^{-}$.

**Lemma 9** *(Continuity properties of concentration functions). Let $s\in(0,\infty)$, $t\in[0,\infty)$, and $Q$ be a probability distribution on $\mathbb{R}$. Then:*

(a) $\mathrm{Conc}^{-}(Q\,;\,t)=\sup_{x\in\mathbb{R}}Q((x,\,x+t))=\sup_{x\in\mathbb{R}}Q([x,\,x+t))$.

(b) $\mathrm{Conc}(Q\,;\,\cdot)$ *is continuous from the right;* $\mathrm{Conc}^{-}(Q\,;\,\cdot)$ *is continuous from the left.*

(c) $\mathrm{Conc}(Q\,;\,s-)=\mathrm{Conc}^{-}(Q\,;\,s)$ *and* $\mathrm{Conc}^{-}(Q\,;\,t+)=\mathrm{Conc}(Q\,;\,t)$.

(d) *There exists an $x_t\in\mathbb{R}$ such that* $\mathrm{Conc}(Q\,;\,t)=Q([x_t,\,x_t+t])$.

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# References


[1] Arak, T.V. and Zaĭtsev, A.Yu. (1988) *Uniform Limit Theorems for Sums of Independent Random Variables. Transl. from Russian.* (English). Proc. Steklov Inst. Math., **174**(1), 222 pp., Providence, RI: American Mathematical Society (AMS).

[2] Barbour, A.D., Chen, L.H.Y. and Loh, W.-L. (1992) Compound Poisson approximation for nonnegative random variables via Stein's method. *Ann. Probab.*, **20**, 1843–1866.

[3] Barbour, A.D., Holst, L. and Janson, S. (1992) *Poisson Approximation.* Oxford: Clarendon Press.

[4] Barbour, A.D. and Xia, A. (1999) Poisson perturbations. *ESAIM: Probab. Statist.*, **3**, 131–150.

[5] Barbour, A.D. and Xia, A. (2000) Estimating Stein's constants for compound Poisson approximation. *Bernoulli*, **6**, 581–590.

[6] Bening, V.E., Korolev, V.Yu. and Shorgin, S.Ya. (1997) On approximations to generalized Poisson distributions. *J. Math. Sci., New York*, **83**, 360–373.

[7] Čekanavičius, V. (1997) Approximation of the generalized Poisson binomial distribution: Asymptotic expansions. *Lith. Math. J.*, **37**, 1–12.

[8] Čekanavičius, V. (2003) Infinitely divisible approximations for discrete nonlattice variables. *Adv. Appl. Probab.*, **35**, 982–1006.

[9] Feller, W. (1971) *An Introduction to Probability Theory and Its Applications.* Vol. II, 2nd ed., New York: John Wiley.

[10] Hengartner, W. and Theodorescu, R. (1973) *Concentration Functions.* New York: Academic Press.

[11] Hipp, C. (1985) Approximation of aggregate claims distributions by compound Poisson distributions. *Insur. Math. Econ.*, **4**, 227–232. Correction note: **6**, 165, (1987).

[12] Hipp, C. (1986) Improved approximations for the aggregate claims distribution in the individual model. *ASTIN Bull.*, **16**, 89–100.

[13] Hipp, C. and Michel, R. (1990) *Risikotheorie: Stochastische Modelle und Statistische Verfahren.* Karlsruhe: Verlag Versicherungswirtschaft.

[14] Kesten, H. (1969) A sharper form of the Doeblin-Lévy-Kolmogorov-Rogozin inequality for concentration functions. *Math. Scand.*, **25**, 133–144.

[15] Kolmogorov, A.N. (1956) Two uniform limit theorems for sums of independent random variables. *Theory Probab. Appl.*, **1**, 384–394.

[16] Kolmogorov, A. (1958) Sur les propriétés des fonctions de concentrations de M.P. Lévy. (French) *Ann. Inst. Henri Poincaré*, **16**, 27–34.

[17] Kornya, P.S. (1983) Distribution of aggregate claims in the individual risk theory model. *Transactions Soc. Actuaries*, **35**, 823–858.

[18] Kruopis, J. (1986) Precision of approximation of the generalized binomial distribution by convolutions of Poisson measures. *Lith. Math. J.*, **26**, 37–49.

[19] Le Cam, L. (1965) On the distribution of sums of independent random variables. In J. Neyman and L. Le Cam (eds), *Bernoulli, Bayes, Laplace. Anniversary volume. Proceedings of an international research seminar, Statistical Laboratory, University of California, Berkeley 1963*, pp. 179–202, New York: Springer.

[20] Le Cam, L. (1986) *Asymptotic Methods in Statistical Decision Theory.* New York: Springer.

[21] Müller, A. and Stoyan, D. (2002) *Comparison Methods for Stochastic Models and Risks.* Chichester: John Wiley.

[22] Nagaev, S.V. and Khodzhibagyan, S.S. (1996) On an estimate of the concentration function of sums of independent random variables. *Theory Probab. Appl.*, **41**, 560–569.

[23] Panjer, H.P. (1981) Recursive evaluation of a family of compound distributions. *ASTIN Bull.*, **12**, 22–26.

[24] Petrov, V.V. (1975) *Sums of Independent Random Variables.* Berlin: Springer.

[25] Petrov, V.V. (1995) *Limit Theorems of Probability Theory: Sequences of Independent Random Variables.* Oxford: Clarendon Press.

[26] Presman, É.L. (1983) Approximation of binomial distributions by infinitely divisible ones. *Theory Probab. Appl.*, **28**, 393–403.

[27] Rogozin, B.A. (1961) An estimate for concentration functions. *Theory Probab. Appl.*, **6**, 94–97.

[28] Roos, B. (2001) Sharp constants in the Poisson approximation. *Statist. Probab. Lett.*, **52**, 155–168.

[29] Roos, B. (2002) Kerstan's method in the multivariate Poisson approximation: an expansion in the exponent. *Theory Probab. Appl.*, **47**, 358–363.

[30] Roos, B. (2003) Kerstan's method for compound Poisson approximation. *Ann. Probab.*, **31**, 1754–1771.

[31] Salikhov, N.P. (1996) An estimate of the concentration function by the Esseen method. *Theory Probab. Appl.*, **41**, 504–518.

[32] Zaĭtsev, A.Yu. (1983) On the accuracy of approximation of distributions of sums of independent random variables – which are nonzero with a small probability – by means of accompanying laws. *Theory Probab. Appl.*, **28**, 657–669.