An Expansion in the Exponent for Compound Binomial Approximations

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Abstract

The purpose of this paper is two-fold: First, we introduce a new asymptotic expansion in the exponent for the compound binomial approximation of the generalized Poisson binomial distribution. The dependence of its accuracy on the symmetry and shifting of distributions is investigated. Second, for compound binomial and compound Poisson distributions, we present new smoothness estimates, some of which contain explicit constants. Finally, the ideas used in this paper enable us to prove new precise bounds in the compound Poisson approximation.

Keywords: Compound binomial distribution, compound Poisson distribution, expansion in the exponent, Kolmogorov norm, concentration norm, sharp constants, shifted distributions, symmetric distributions, total variation norm.

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1 Introduction

1.1 Aim of the paper

In principle, the investigation of signed approximations with exponential structure started in the early 1980s, when Kornya [22] introduced signed compound Poisson approximations. Kornya's approximation was based on an asymptotic expansion in the exponent for probability distributions having positive mass at zero. Independently, Presman [26] used a signed exponential measure for the approximation of the binomial distribution. Generally speaking, Presman's approximation was constructed taking into account the logarithmic series for the nth power of a characteristic function $f^n(t)$, $(t \in \mathbb{R}$, $n \in \mathbb{N} = \{1, 2, \dots\})$ of a probability distribution F defined on the real line $\mathbb{R}$. In fact, $f^n(t)$ can be expanded in the following way:

$$f^n(t) = \mathrm{e}^{n \log f(t)} = \exp\Big\{n(f(t)-1) - \frac{n}{2}(f(t)-1)^2 + \frac{n}{3}(f(t)-1)^3 - \dots\Big\}. \qquad (1)$$

A signed compound Poisson approximation of the n-fold convolution of F can then be constructed by choosing a finite signed measure with the Fourier transform similar as in the right-hand side of (1) but with only the first $s \in \mathbb{N}$ summands in the exponent.

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The following years were marked by a growing interest in signed approximations, which can be derived from expansions in the exponent. Considering signed compound Poisson approximations, Hipp [19] proved that estimates for the total variation distance hold under very general conditions (see (7) below). Kruopis [23] proposed to apply approximations based on the expansion in factorial cumulants. Borovkov and Pfeifer [6] considered an approximation for Markov chains. Barbour and Xia [4] showed how Stein's method can be used for signed compound Poisson approximations. In the lattice case, Panjer-type recursive algorithms for calculational purposes were developed by Dhaene and De Pril [14].

Note that all authors mentioned above considered approximations, which can be derived from expansions in the exponent taking into account the compound Poisson distribution. There seem to be only a few publications treating exponential expansions for other probability distributions. One paper in this direction was published by Bikelis [5], who proposed an expansion, which involves an arbitrary characteristic function $g(t)$. It reads as:

$$f^n(t) = g^n(t)\,\mathrm{e}^{n\,\log(1+h(t))} = g^n(t)\exp\{n\,h(t)\}\Big(1-\frac{n}{2}h^2(t)+\frac{n}{3}h^3(t)+\dots\Big), \qquad (2)$$

where $h(t) = f(t)/g(t) - 1$. Further, in Čekanavičius [8], an exponential expansion in the context of the normal distribution was introduced. It is evident that expansions as in (1) and (2) are also possible for the convolution product of not necessarily identical distributions.

In this paper, we use a compound binomial type expansion of the generalized Poisson binomial distribution, which is comparable with (2). In principle, the main difference is that the whole expansion is in the exponent. Note that, though the construction of our approximations is based on simple ideas, estimating of their accuracy usually is far from being trivial.

The structure of the paper is the following. In the next two subsections, we proceed with some notation and discussion of important known facts. Section 2 is then devoted to the main results. Here, bounds for the approximation error in different distances are given and, under special assumptions, bounds with asymptotically sharp constants are presented. As a byproduct, using the ideas of this paper, we obtain some precise bounds in the compound Poisson approximation. A significant part of the paper is devoted to the refinements of some auxiliary smoothness estimates for the compound binomial and compound Poisson distributions (see Section 3). Sometimes the estimates are supplied with explicit constants; in other cases, weaker assumptions on the parameters are used. In Section 4, we give the proofs of the main results. We note that our aim is to get our constants as precise as possible, as a consequence of which some proofs are quite long and elaborate.

1.2 Notation

Let $\mathcal{F}$ (resp. $\mathcal{S}$, resp. $\mathcal{M}$) denote the set of all probability distributions (resp. symmetric probability distributions about zero, resp. finite signed measures) on $\mathbb{R}$. All products and powers of finite signed measures in $\mathcal{M}$ are defined in the convolution sense; for $W \in \mathcal{M}$, set $W^0 = I = I_0$, where I_u is the Dirac measure at point $u \in \mathbb{R}$. The exponential of W is defined by the finite signed measure

$$\exp\{W\} = \sum_{m=0}^{\infty} \frac{W^m}{m!}.$$

The compound Poisson distribution with parameters $t \in [0,\infty)$ and $F \in \mathcal{F}$ is given by $\exp\{t(F-I)\}$. Let $W = W^+ - W^-$ denote the Hahn–Jordan decomposition of W. The total variation norm $\|W\|$, the Kolmogorov norm $|W|$, and the Lévy concentration norm $|W|_h$ of $W \in \mathcal{M}$ are defined by

$$\begin{aligned}\|W\| &= W^+(\mathbb{R}) + W^-(\mathbb{R}),\\ |W| &= \sup_{x\in\mathbb{R}} |W((-\infty, x])|,\\ |W|_h &= \sup_{x\in\mathbb{R}} |W([x, x+h])|, \qquad (h \in [0,\infty)),\end{aligned}$$

respectively. It should be mentioned that $|\cdot|_0$ is only a seminorm on $\mathcal{M}$, i.e. it may happen that, for non-zero $W \in \mathcal{M}$, $|W|_0 = 0$. But if we restrict ourselves to finite signed measures concentrated on the set of all integers $\mathbb{Z}$, then $|\cdot|_0$ is indeed a norm, the so-called local norm. For $h \in (0,\infty)$ and a finite measure G on $\mathbb{R}$, set $|G|_{h-} = \lim_{y\uparrow h} |G|_y$. It is well-known that

$$|G|_{h-} = \sup_{x\in\mathbb{R}} G((x, x+h]) = \sup_{x\in\mathbb{R}} G([x, x+h)) = \sup_{x\in\mathbb{R}} G((x, x+h)),$$

see Hengartner and Theodorescu [18]. Note that, here, it is essential to assume that G is a non-negative measure. Set $|G|_{0-} = 0$. For $W \in \mathcal{M}$ and a power series $g(z) = \sum_{m=0}^{\infty} a_m z^m$, ($a_m \in \mathbb{R}$), converging absolutely for each complex $z \in \mathbb{C}$ with $|z| \leq \|W\|$, we define $g(W) = \sum_{m=0}^{\infty} a_m W^m \in \mathcal{M}$. In the paper, we will often use the well-known relations

$$\|VW\| \leq \|V\|\,\|W\|, \qquad |VW| \leq |V|\,\|W\|, \qquad |VW|_h \leq |V|_h\,\|W\|,$$
$$\max\{|W|,\, |W|_h\} \leq \|W\|,$$

for $V, W \in \mathcal{M}$, $h \in [0,\,\infty)$, and

$$\max\{|W|,\, |W|_h\} \leq \frac{1}{2}\,\|W\|, \tag{3}$$

for $W \in \mathcal{M}$ with $W(\mathbb{R}) = 0$ and $h \in [0,\infty)$. We denote by C positive absolute constants, the values of which may change from line to line, or even within the same line. Similarly, by $C(\cdot)$ we denote constants depending on the indicated argument only. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the largest integer not exceeding x. Always, let $0^0 = 1$, $1/0 = \infty$, and, for $k \in \mathbb{Z}$, $\sum_{m=k}^{k-1} = 0$ be the empty sum. For two real valued functions f and g defined on some subset of $\mathbb{R}$, $f(x) \sim g(x)$, ($x \to a \in \mathbb{R}\cup\{\pm\infty\}$) means that $\lim_{x\to a} f(x)/g(x) = 1$. Set

$$\varphi_0(x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}, \quad \varphi_k(x) = \frac{\mathrm{d}^k}{\mathrm{d}x^k}\varphi_0(x), \quad (k \in \mathbb{N},\ x \in \mathbb{R}), \tag{4}$$
$$\|\varphi_k\|_1 = \int_{\mathbb{R}} |\varphi_k(x)|\,\mathrm{d}x, \quad \|\varphi_k\|_\infty = \sup_{x\in\mathbb{R}} |\varphi_k(x)|, \quad (k \in \mathbb{Z}_+ = \mathbb{N}\cup\{0\}).$$

Let $n \in \mathbb{N}$ and

$$p_j = 1 - q_j \in [0,1], \quad (j \in \{1,\dots,n\}), \quad \mathbf{p} = (p_1,\dots,p_n), \quad \lambda_k = \sum_{j=1}^n p_j^k, \quad (k \in \mathbb{N}),$$
$$\lambda = \lambda_1 > 0, \quad \overline{p} = \frac{\lambda}{n}, \quad \overline{q} = 1 - \overline{p}, \quad p_{\max} = \max_{1\leq j\leq n} p_j, \quad p_{\min} = \min_{1\leq j\leq n} p_j,$$
$$\delta = p_{\max} - p_{\min}, \quad \theta = \frac{\lambda_2}{\lambda}, \quad \gamma_k = \sum_{j=1}^n (\overline{p} - p_j)^k, \quad \nu_k = \sum_{j=1}^n |\overline{p} - p_j|^k, \quad (k \in \mathbb{N}).$$

The generalized Poisson binomial and compound binomial distributions are denoted by

$$\mathrm{GPB}(n,\,\mathbf{p},\,F)=\prod_{j=1}^{n}(q_jI+p_jF),\qquad \mathrm{Bi}(n,\,p,\,F)=(qI+pF)^n,$$

where $F\in\mathcal{F}$ and $p=1-q\in[0,1]$. The binomial distribution with parameters $n\in\mathbb{N}$ and $p\in[0,1]$ is then given by $\mathrm{Bi}(n,\,p)=\mathrm{Bi}(n,\,p,\,I_1)$. Let $s\in\mathbb{N}$ and, for $F\in\mathcal{F}$ and $j\in\{1,\dots,n\}$, set

$$\begin{aligned}
U_j=U_j(n,\,\mathbf{p},\,F) &= (p_j-\overline{p})(F-I)\sum_{k=0}^{\infty}(-\overline{p})^k(F-I)^k,\\
D=D(n,\,\mathbf{p},\,F;\,s) &= \exp\Big\{\sum_{j=1}^{n}\sum_{m=2}^{s}\frac{(-1)^{m+1}}{m}U_j^m\Big\},\\
\mathrm{EBi}(n,\,\mathbf{p},\,F;\,s) &= D(n,\,\mathbf{p},\,F;\,s)(\overline{q}I+\overline{p}F)^n.
\end{aligned}$$

Whenever we deal with U_j or D, we assume that $\overline{p}<1/2$, and, therefore, U_j and D are elements of $\mathcal{M}$ with finite total variation norms

$$\|U_j\|\le\frac{2|p_j-\overline{p}|}{1-2\overline{p}}\le\frac{2\delta}{1-2\overline{p}},\qquad \|D\|\le\exp\Big\{\sum_{j=1}^{n}\sum_{m=2}^{s}\frac{1}{m}\|U_j\|^m\Big\}. \tag{5}$$

Note that, if $\delta+\overline{p}<1/2$, then $D(n,\,\mathbf{p},\,F;\,\infty)\in\mathcal{M}$ can be defined as above and we get, by using characteristic functions,

$$\mathrm{GPB}(n,\,\mathbf{p},\,F)=D(n,\,\mathbf{p},\,F;\,\infty)(\overline{q}I+\overline{p}F)^n. \tag{6}$$

This equality motivates the approximation of $\mathrm{GPB}(n,\,\mathbf{p},\,F)$ by $\mathrm{EBi}(n,\,\mathbf{p},\,F;\,s)$. Observe that $\mathrm{EBi}(n,\,\mathbf{p},\,F;\,s)$ can be viewed as a function of $(n,\overline{p},s,\gamma_1,\dots,\gamma_s,F)$ and is, in this regard, of a simpler form than $\mathrm{GPB}(n,\,\mathbf{p},\,F)$, at least when s is small. In Section 2 below, it turns out that, as we should expect, the accuracy of our error bounds is increasing in s.

1.3 Some known results

The compound Poisson approximation of the distribution of the sum of independent random variables was considered in numerous publications; for example, see Le Cam [24, 25], Arak and Zaĭtsev [1], Čekanavičius [9], Barbour and Chryssaphinou [2], Roos [32], and the references therein. One of the most general results, involving approximations derived from an asymptotic expansion in the exponent, was obtained by Hipp (see formula (6) in [19]), who proved that, for $p_j\in[0,1/2)$, $F_j\in\mathcal{F}$, $(j\in\{1,\dots,n\})$, and $s\in\mathbb{N}$,

$$\Big\|\prod_{j=1}^{n}(q_jI+p_jF_j)-\exp\Big\{\sum_{j=1}^{n}\sum_{m=1}^{s}\frac{(-1)^{m+1}}{m}p_j^m(F_j-I)^m\Big\}\Big\|\le\exp\Big\{\frac{2^{s+1}}{s+1}\sum_{j=1}^{n}\frac{p_j^{s+1}}{1-2p_j}\Big\}-1. \tag{7}$$

Binomial and compound binomial approximations were studied not so comprehensively. Ehm ([15], Theorem 1 and Lemma 2) proved that the total variation distance between $\mathrm{GPB}(n,\,\mathbf{p},\,I_1)$ and the binomial distribution $\mathrm{Bi}(n,\,\overline{p})$ can be estimated in the following way:

$$\frac{\gamma_2}{62}\min\Big\{1,\,\frac{1}{n\overline{p}\,\overline{q}}\Big\}\le\|\mathrm{GPB}(n,\,\mathbf{p},\,I_1)-\mathrm{Bi}(n,\overline{p})\|\le2\gamma_2\min\Big\{1,\,\frac{1}{n\overline{p}\,\overline{q}}\Big\}. \tag{8}$$

As can be seen from (8), the estimate depends on the behavior of the so-called magic factor $(n\overline{p}\,\overline{q})^{-1}$ (cf. Introduction in Barbour *et al.* [3]), and on the closeness of all p_j. The last fact is reflected by γ_2. In Roos [29], an approximation $\mathrm{Bi}(n,\,p,\,I_1;\,s)$ with a general $p \in [0,1]$ based on an expansion in Krawtchouk polynomials was constructed. We note that, in [29], this signed measure was denoted by $\mathcal{B}_s(n,p)$. By inserting a general $F \in \mathcal{F}$ into the generating function of $\mathrm{Bi}(n,\,p,\,I_1;\,s)$, we obtain the signed measure $\mathrm{Bi}(n,\,p,\,F;\,s)$. In view of the properties of the total variation distance, the case of arbitrary $F \in \mathcal{F}$ can be reduced to the case $F = I_1$ and, therefore, one of the main results from Roos (see Theorem 2 in [29] or Corollary 1 in [31]) can be written as:

$$\begin{aligned}\sup_{F\in\mathcal{F}} \|\mathrm{GPB}(n,\,\mathbf{p},\,F) - \mathrm{Bi}(n,\,\overline{p},\,F;\,s)\| &= \|\mathrm{GPB}(n,\,\mathbf{p},\,I_1) - \mathrm{Bi}(n,\,\overline{p},\,I_1;\,s)\| \\ &\le C(s)\Big(\gamma_2 \min\Big\{1,\,\frac{1}{n\overline{p}\,\overline{q}}\Big\}\Big)^{(s+1)/2}. \qquad (9)\end{aligned}$$

Note that the case $s = 1$ corresponds to Ehm's approximation. The compound binomial approximation of the generalized Poisson binomial distribution was further investigated in Čekanavičius and Roos [12]. It was shown that the possible shifting or symmetry of F provide better results. In particular, from Theorem 3.1 together with Remark 1(ii) of that paper it follows that, if $p_{\max} \le 0.3$, then

$$\sup_{F\in\mathcal{F}} \inf_{u\in\mathbb{R}} |\mathrm{GPB}(n,\,\mathbf{p},\,I_uF) - \mathrm{Bi}(n,\,\overline{p},\,I_uF;\,s)| \le C(s)\,\frac{\gamma_2^{(s+1)/2}}{\lambda^{(s+1)/2+(s+1)/(2s+4)}}, \qquad (10)$$

$$\sup_{F\in\mathcal{S}} |\mathrm{GPB}(n,\,\mathbf{p},\,F) - \mathrm{Bi}(n,\,\overline{p},\,F;\,s)| \le C(s)\,\frac{\gamma_2^{(s+1)/2}}{\lambda^{s+1}}. \qquad (11)$$

In the present paper, we prove that the approximation by the signed measures $\mathrm{EBi}(n,\,\mathbf{p},\,F;\,s)$ derived from the expansion in the exponent, is more accurate than (9)–(11). First of all, the bounds much better reflect the closeness of p_j. For example, it is shown with (14) below, that, in the general case $F \in \mathcal{F}$, under the assumption that $p_{\max} \le 1/5$, the accuracy can be estimated by $C(s)\,\nu_{s+1}\lambda^{-(s+1)/2}$. If λ is large and if $s \ge 2$, then this bound is small, since generally

$$\frac{\gamma_2}{\lambda\,\overline{q}} \le \delta \min\Big\{1,\,\frac{\delta}{4\overline{p}\,\overline{q}}\Big\} \qquad (12)$$

(see Remark on p. 259 in [29]) and therefore $\nu_{s+1} \le \delta^{s-1}\,\gamma_2 \le \delta^s\,\lambda\overline{q}$. In this sense, (14) provides a much better bound than the one of (9). In the cases of a shifted or symmetric distribution F, the improvement in the accuracy is similar.

Note that the accuracy of approximation in (8) and (9) is estimated by a minimum of two quantities, one of which containing a magic factor. As a rule, the estimate with the magic factor is much more difficult to prove than the other one. Sometimes, for this case, more restrictive assumptions are used. Indeed, in (13) below, we prove an analogue of Hipp's result (see (7)) for the compound binomial approximation, which does not exhibit a magic factor, but requires only the assumption that $\delta + \overline{p} < 1/2$. In contrast to this, most of the other main results are proved under the stronger condition $p_{\max} \le 1/5$. Moreover, due to (3), the total variation bound in (13) can also be used for the concentration and Kolmogorov norms. Therefore, it is clear that each estimate in Theorems 2.2 and 2.3 below, which contains some magic factor, can be rewritten, in the spirit of (8), as the minimum of the bound in (13) and the corresponding estimate.

It should be mentioned that it is possible to choose the parameters of the underlying compound binomial distribution in a way different from the one of this paper. In fact, a compound binomial distribution $\mathrm{Bi}(N,\, \widetilde{p},\, F)$ can be used, where the two parameters $N \in \mathbb{N}$ and $\widetilde{p} \in [0, 1]$ are chosen in such a way that, in the case $F = I_1$, both distributions $\mathrm{Bi}(N,\, \widetilde{p},\, F)$ and $\mathrm{GPB}(n,\, \mathbf{p},\, F)$ have the same mean and approximately the same variance. This two-parametric binomial approach was used in Barbour *et al.* ([3], p. 190) and Soon [34] for the binomial approximation of the Poisson binomial distribution. In Čekanavičius and Roos [11], it was extended to the compound binomial case. Generally speaking, in comparison with the approach of the present paper, this two-parameter choice implies better magic factors. However, certain difficulties arise in the construction of asymptotic expansions, since N must be integer valued. Moreover, in comparison to the results below, the bounds of the two-parametric compound binomial approximation much weaker reflect the possible closeness of the p_j. We note that the main results of this paper can be applied to the two-parametric compound binomial approximation, see Theorem 2.6 below.

2 Main results

Our first result is a Hipp-type bound without any magic factor (cf. (7)).

Theorem 2.1 *Let $\delta + \overline{p} < 1/2$ and $F \in \mathcal{F}$. Then*

$$\|\mathrm{GPB}(n,\, \mathbf{p},\, F) - \mathrm{EBi}(n,\, \mathbf{p},\, F;\, s)\| \le \exp\Big\{\frac{2^{s+1}}{(s+1)(1-2\overline{p})^s}\sum_{j=1}^{n}\frac{|p_j - \overline{p}|^{s+1}}{1 - 2(|p_j - \overline{p}| + \overline{p})}\Big\} - 1. \tag{13}$$

It turns out that the bound in the previous theorem is of order ν_{s+1}. More precisely we have the following corollary.

Corollary 2.1 *Let $\delta + \overline{p} \le C < 1/2$, $\nu_{s+1} \le C$, and $F \in \mathcal{F}$. Then*

$$\|\mathrm{GPB}(n,\, \mathbf{p},\, F) - \mathrm{EBi}(n,\, \mathbf{p},\, F;\, s)\| \le C(s)\,\nu_{s+1}.$$

In the next theorem, we present some bounds with magic factors. As already discussed in the introduction, in contrast to the general case $F \in \mathcal{F}$, the shifting of F or the assumption $F \in \mathcal{S}$ improves the order. In fact, the exponent of $1/\lambda$ can be increased.

Theorem 2.2 *Let us assume that $p_{\max} \le 1/5$. For $F \in \mathcal{F}$, we then have*

$$\begin{aligned}
\Big\|\mathrm{GPB}(n,\, \mathbf{p},\, F) - \mathrm{EBi}(n,\, \mathbf{p},\, F;\, s)\Big\| &\le C(s)\,\frac{\nu_{s+1}}{\lambda^{(s+1)/2}}, &(14)\\
\inf_{u\in\mathbb{R}}\Big|\mathrm{GPB}(n,\, \mathbf{p},\, I_uF) - \mathrm{EBi}(n,\, \mathbf{p},\, I_uF;\, s)\Big| &\le C(s)\,\frac{\nu_{s+1}}{\lambda^{(s+1)/2+(s+1)/(2s+4)}}. &(15)
\end{aligned}$$

For $F \in \mathcal{S}$ and $h \in [0, \infty)$, we have

$$\begin{aligned}
\Big|\mathrm{GPB}(n,\, \mathbf{p},\, F) - \mathrm{EBi}(n,\, \mathbf{p},\, F;\, s)\Big| &\le C(s)\,\frac{\nu_{s+1}}{\lambda^{s+1}}, &(16)\\
\Big|\mathrm{GPB}(n,\, \mathbf{p},\, F) - \mathrm{EBi}(n,\, \mathbf{p},\, F;\, s)\Big|_h &\le C(s)\,\frac{\nu_{s+1}}{\lambda^{s+1}}\,Q_h^{1/(2s+5)}\\
&\quad\times(|\ln Q_h| + 1)^{6(s+2)(s+3)/(2s+5)}, &(17)
\end{aligned}$$

where $Q_h := Q_{h,\lambda\overline{q},F} := |\exp\{40^{-1}\lambda\overline{q}(F - I)\}|_h$.

Remark 2.1 (a) If $p_j = \overline{p}$ for all j, then $\mathrm{GPB}(n,\,\mathbf{p},\,F)$ and $\mathrm{EBi}(n,\,\mathbf{p},\,F;\,s)$ coincide and this is reflected in the bounds (13)–(17).

(b) In the definition of $\mathrm{EBi}(n,\,\mathbf{p},\,F;\,s)$, we have to assume that $\overline{p} < 1/2$. Therefore, in our results, the assumption on $p_{\max}$ cannot be entirely dropped.

(c) It is easily shown that (16) follows from (17). However, (17) leads to an estimate of a better order than the one in (16), if we use a Le Cam-type bound for the concentration function of compound Poisson distributions; for example, see Proposition 3 in Roos [32], where it was shown that, for $t \in (0,\infty)$, $h \in [0,\infty)$, and an arbitrary distribution $F \in \mathcal{F}$,

$$|\exp\{t(F-I)\}|_h \le \frac{1}{\sqrt{2\mathrm{e}\,t\max\{F((-\infty,-h)),\,F((h,\infty))\}}}. \tag{18}$$

Generally, for the total variation distance, there seems to be no hope for upper bounds similar to (15) and (16). However, for symmetric distributions concentrated on the set of integers, the following theorem can be shown.

Theorem 2.3 *Let $p_{\max} \le 1/5$, $h \in [0,\infty)$, and $F \in \mathcal{S}$ be concentrated on the set $\mathbb{Z}\setminus\{0\}$. Then*

$$\|\mathrm{GPB}(n,\,\mathbf{p},\,F) - \mathrm{EBi}(n,\,\mathbf{p},\,F;\,s)\| \;\le\; C(s)\,\sqrt{\sigma}\,\frac{\nu_{s+1}}{\lambda^{s+1}}, \tag{19}$$

$$|\mathrm{GPB}(n,\,\mathbf{p},\,F) - \mathrm{EBi}(n,\,\mathbf{p},\,F;\,s)|_h \;\le\; C(s)\,\lfloor h+1\rfloor\,\frac{\nu_{s+1}}{\lambda^{s+3/2}}, \tag{20}$$

where, for (19), *we assume that F has finite variance σ^2.*

Remark 2.2 (a) The total variation bound (19) is slightly worse than the one in (16), since the variance σ^2 of the $F \in \mathcal{S}$ concentrated on $\mathbb{Z}\setminus\{0\}$ cannot be smaller than one.

(b) Inequality (20) has a better order than the bound, which can be derived from (17) and (18).

In the previous results, the leading constants $C(s)$ are usually not given explicitly, since, due to the method of proof, they appear to be quite large. However, in the special case when F is a Dirac measure or symmetric distribution concentrated on two points, some asymptotically sharp constants can be evaluated.

Theorem 2.4 *Let $p_{\max} \le 1/5$ and set*

$$c_{s+1}^{(1)} = \frac{\|\varphi_{s+1}\|_1}{s+1}, \qquad c_{s+1}^{(2)} = \frac{\|\varphi_{s}\|_\infty}{s+1}, \qquad c_{s+1}^{(3)} = \frac{\|\varphi_{s+1}\|_\infty}{s+1}.$$

Then

$$\Big|\|\mathrm{GPB}(n,\,\mathbf{p},\,I_1) - \mathrm{EBi}(n,\,\mathbf{p},\,I_1;\,s)\| - \frac{c_{s+1}^{(1)}\,|\gamma_{s+1}|}{(\lambda\,\overline{q})^{(s+1)/2}}\Big| \;\le\; \frac{C(s)\,\nu_{s+1}}{\lambda^{(s+1)/2}}\Big(\frac{1}{\sqrt{\lambda}} + \frac{\gamma_2}{\lambda}\Big), \tag{21}$$

$$\Big||\mathrm{GPB}(n,\,\mathbf{p},\,I_1) - \mathrm{EBi}(n,\,\mathbf{p},\,I_1;\,s)| - \frac{c_{s+1}^{(2)}\,|\gamma_{s+1}|}{(\lambda\,\overline{q})^{(s+1)/2}}\Big| \;\le\; \frac{C(s)\,\nu_{s+1}}{\lambda^{(s+1)/2}}\Big(\frac{1}{\sqrt{\lambda}} + \frac{\gamma_2}{\lambda}\Big), \tag{22}$$

$$\Big||\mathrm{GPB}(n,\,\mathbf{p},\,I_1) - \mathrm{EBi}(n,\,\mathbf{p},\,I_1;\,s)|_0 - \frac{c_{s+1}^{(3)}\,|\gamma_{s+1}|}{(\lambda\,\overline{q})^{(s+2)/2}}\Big| \;\le\; \frac{C(s)\,\nu_{s+1}}{\lambda^{(s+2)/2}}\Big(\frac{1}{\sqrt{\lambda}} + \frac{\gamma_2}{\lambda}\Big). \tag{23}$$

Theorem 2.5 *Let $p_{\max} \le 1/5$ and $F = 2^{-1}(I_{-1} + I_1)$. Set*

$$c_{s+1}^{(4)} = \frac{\|\varphi_{2s+2}\|_1}{(s+1)\,2^{s+1}}, \qquad c_{s+1}^{(5)} = \frac{\|\varphi_{2s+1}\|_\infty}{(s+1)\,2^{s+1}}, \qquad c_{s+1}^{(6)} = \frac{\|\varphi_{2s+2}\|_\infty}{(s+1)\,2^{s+1}}.$$

Then

$$\left|\|\mathrm{GPB}(n,\,\mathbf{p},\,F) - \mathrm{EBi}(n,\,\mathbf{p},\,F;\,s)\| - \frac{c_{s+1}^{(4)}\,|\gamma_{s+1}|}{\lambda^{s+1}}\right| \le C(s)\,\frac{\nu_{s+1}}{\lambda^{s+3/2}}, \tag{24}$$

$$\left||\mathrm{GPB}(n,\,\mathbf{p},\,F) - \mathrm{EBi}(n,\,\mathbf{p},\,F;\,s)| - \frac{c_{s+1}^{(5)}\,|\gamma_{s+1}|}{\lambda^{s+1}}\right| \le C(s)\,\frac{\nu_{s+1}}{\lambda^{s+3/2}}, \tag{25}$$

$$\left||\mathrm{GPB}(n,\,\mathbf{p},\,F) - \mathrm{EBi}(n,\,\mathbf{p},\,F;\,s)|_0 - \frac{c_{s+1}^{(6)}\,|\gamma_{s+1}|}{\lambda^{s+3/2}}\right| \le C(s)\,\frac{\nu_{s+1}}{\lambda^{s+2}}. \tag{26}$$

Remark 2.3 The constants $c_{s+1}^{(1)}, \dots, c_{s+1}^{(6)}$ in Theorems 2.4 and 2.5 can be evaluated for small $s \in \mathbb{N}$ by using the following identities, which are not difficult to prove:

$$\|\varphi_1\|_1 = \sqrt{\frac{2}{\pi}}, \quad \|\varphi_1\|_\infty = \frac{1}{\sqrt{2\pi\mathrm{e}}}, \quad \|\varphi_2\|_1 = \frac{4}{\sqrt{2\pi\mathrm{e}}}, \quad \|\varphi_2\|_\infty = \frac{1}{\sqrt{2\pi}},$$
$$\|\varphi_3\|_1 = \sqrt{\frac{2}{\pi}}(1 + 4\mathrm{e}^{-3/2}), \quad \|\varphi_3\|_\infty = \sqrt{\frac{3}{\pi}}\exp\Big\{\sqrt{\frac{3}{2}} - \frac{3}{2}\Big\}\sqrt{3-\sqrt{6}},$$
$$\|\varphi_4\|_1 = 4\mathrm{e}^{-3/2}\sqrt{\frac{3}{\pi}}\Big[\exp\Big\{\sqrt{\frac{3}{2}}\Big\}\sqrt{3-\sqrt{6}} + \exp\Big\{-\sqrt{\frac{3}{2}}\Big\}\sqrt{3+\sqrt{6}}\Big],$$
$$\|\varphi_4\|_\infty = \frac{3}{\sqrt{2\pi}}, \quad \|\varphi_5\|_1 = \frac{2(3\mathrm{e}^{5/2} - 32\sinh(\sqrt{5/2}) + 16\sqrt{10}\cosh(\sqrt{5/2}))}{\sqrt{2\pi}\,\mathrm{e}^{5/2}}.$$

Remark 2.4 (a) Assume that

$$p_{\max} \le \frac{1}{5}, \qquad \nu_{s+1} \le C\,|\gamma_{s+1}|, \qquad \lambda \to \infty, \tag{27}$$

and that $\gamma_2/\lambda \to 0$. Then (21) yields

$$\|\mathrm{GPB}(n,\,\mathbf{p},\,I_1) - \mathrm{EBi}(n,\,\mathbf{p},\,I_1;\,s)\| \sim \frac{c_{s+1}^{(1)}|\gamma_{s+1}|}{(\lambda\,\overline{q})^{(s+1)/2}}.$$

Similar assertions follow from (22) and (23). Note that the conditions above are not very restrictive. For example, they are valid, if $0 < C \le p_{\min} \le p_{\max} \le 1/5$, s is odd, $n \to \infty$, $\delta \to 0$, (see (12)).

(b) Assume now that $F = 2^{-1}(I_{-1} + I_1)$ and that the conditions of (27) hold. Then (24) implies that

$$\|\mathrm{GPB}(n,\,\mathbf{p},\,F) - \mathrm{EBi}(n,\,\mathbf{p},\,F;\,s)\| \sim \frac{c_{s+1}^{(4)}\,|\gamma_{s+1}|}{\lambda^{s+1}}.$$

In contrast to the assumptions made in Remark 2.4(a), here, we do not need to suppose that $\gamma_2/\lambda \to 0$. Similar assertions follow from (25) and (26).

(c) The norms in (21)–(26) do not change, if we assume that $F = I_u$ or $F = 2^{-1}(I_{-u}+I_u)$ for some $u \in (0,\infty)$, respectively. Therefore, under these assumptions, Theorems 2.4 and 2.5 remain valid.

In principle, our results above can be used to get upper bounds in the two-parametric compound binomial approximation. We illustrate this for the Kolmogorov norm in the context of symmetric distributions $F \in \mathcal{S}$. As explained at the end of Section 1.3, we choose the binomial parameters in such a way that $\mathrm{Bi}(N, \widetilde{p}, I_1)$ and $\mathrm{GPB}(n, \mathbf{p}, I_1)$ have the same mean and nearly the same variance.

Theorem 2.6 *Let $N = \lambda/\theta - \widetilde{\delta}$, $N \in \mathbb{N}$, $|\widetilde{\delta}| \leq 1/2$, and*

$$\widetilde{p} = \widetilde{p}_1 = \cdots = \widetilde{p}_N = \frac{\lambda}{N}, \quad \widetilde{p}_{N+1} = \cdots = \widetilde{p}_n = 0, \quad \widetilde{\nu}_k = \sum_{j=1}^{n} |\overline{p} - \widetilde{p}_j|^k, \quad (k \in \mathbb{N}).$$

If $p_{\max} \leq 1/5$ and $\widetilde{p} \leq 1/5$, then

$$\sup_{F\in\mathcal{S}} \Big|\mathrm{GPB}(n, \mathbf{p}, F) - \mathrm{Bi}(N, \widetilde{p}, F)\Big| \leq \frac{C}{\lambda^2}\Big(|\nu_2 - \widetilde{\nu}_2| + \frac{\nu_3 + \widetilde{\nu}_3}{\lambda}\Big). \tag{28}$$

Remark 2.5 Let the notation of Theorem 2.6 be valid. Note that $N \leq n$ is such that the $\widetilde{p}_j$ are correctly defined. In Theorem 2.1 of [11], it was shown that, if $p_{\max} \leq 1/4$ and $\lambda \geq 1$ then

$$\sup_{F\in\mathcal{S}} \Big|\mathrm{GPB}(n, \mathbf{p}, F) - \mathrm{Bi}(N, \widetilde{p}, F)\Big| \leq \frac{C}{\lambda^2}\Big(\frac{\theta^2|\widetilde{\delta}|}{1-\widetilde{\delta}\theta/\lambda} + \Big(\frac{\lambda_3}{\lambda} - \theta^2\Big) + \frac{1}{\lambda^2}\sum_{j=1}^{n} p_j|p_j - \theta|\Big). \tag{29}$$

For a comparison of (28) with (29), we first note that, as is easily seen, $\nu_2 - \widetilde{\nu}_2 = -\theta^2\widetilde{\delta}/(1-\widetilde{\delta}\theta/\lambda)$. Further, clearly we have $\nu_3 \leq \delta\nu_2$, whereas $\widetilde{\nu}_3 \in [2(\sqrt{2}-1)\,\overline{p}\,\widetilde{\nu}_2, \widetilde{p}\,\widetilde{\nu}_2]$. The latter relation follows from the fact that, due to $N \leq n$, we have $\widetilde{p} \geq \overline{p}$, and hence

$$\begin{aligned}
2(\sqrt{2}-1)\,\overline{p}\,\widetilde{\nu}_2 &\leq \frac{N\widetilde{p}^2 - n\overline{p}^2}{\widetilde{p}}[(\widetilde{p} - \sqrt{2}\overline{p})^2 + 2(\sqrt{2}-1)\,\widetilde{p}\overline{p}] = N(\widetilde{p}-\overline{p})[(\widetilde{p}-\overline{p})^2 + \overline{p}^2] \\
&= \widetilde{\nu}_3 = N(\widetilde{p}-\overline{p})^3 + (n-N)\overline{p}^3 \leq N(\widetilde{p}-\overline{p})^2\widetilde{p} + (n-N)\overline{p}^2\widetilde{p} = \widetilde{p}(N\widetilde{p}^2 - n\overline{p}^2) = \widetilde{p}\,\widetilde{\nu}_2.
\end{aligned}$$

On the other hand, we have

$$\frac{\lambda_3}{\lambda} - \theta^2 \in \Big[\frac{p_{\min}^2\,\nu_2}{\overline{p}\lambda}, \frac{p_{\max}\,\nu_2}{\lambda}\Big].$$

Indeed, since $\nu_2 = (2n)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}(p_i - p_j)^2$, we have

$$\frac{p_{\min}^2\,\nu_2}{\overline{p}} \leq \frac{1}{2n\overline{p}}\sum_{i=1}^{n}\sum_{j=1}^{n}(p_i - p_j)^2\,p_ip_j = \lambda_3 - \frac{\lambda_2^2}{\lambda} = \sum_{j=1}^{n} p_j(p_j - \overline{p})^2 - \frac{\nu_2^2}{\lambda} \leq p_{\max}\,\nu_2.$$

Combining the estimates above, we see that, under the assumption that $p_{\max} \leq 1/5$, $\widetilde{p} \leq 1/5$, $\lambda \geq 1$, $\widetilde{\delta} = 0$ and that $0 < C < p_{\min}$, the bound in (28) can be estimated by $C\nu_2/\lambda^3$, whereas the one in (29) is not better than $C(\nu_2/\lambda^3 + \lambda^{-2}\sum_{j=1}^{n} p_j|p_j - \theta|)$.

The ideas used in the proofs below imply bounds in the compound Poisson approximation. In the following theorem, we present four examples with explicit constants.

Theorem 2.7 *Let us assume that* $\theta = \lambda_2/\lambda < 1$. *If* $F \in \mathcal{F}$, *then*

$$\|\mathrm{GPB}(n,\, \mathbf{p},\, F) - \exp\{\lambda(F-I)\}\| \;\leq\; \frac{3\,\theta}{2\mathrm{e}\,(1-\sqrt{\theta})^{3/2}}, \tag{30}$$

$$\inf_{u\in\mathbb{R}} |\mathrm{GPB}(n,\, \mathbf{p},\, I_uF) - \exp\{\lambda(I_uF-I)\}| \;\leq\; \frac{11.2\,\theta}{\lambda^{1/3}\,(1-\sqrt{\theta})^{11/6}}. \tag{31}$$

If $F \in \mathcal{S}$ *is concentrated on* $\mathbb{Z}\setminus\{0\}$ *and* $h \in [0,\infty)$, *then*

$$\|\mathrm{GPB}(n,\, \mathbf{p},\, F) - \exp\{\lambda(F-I)\}\| \;\leq\; \frac{1.16\sqrt{\sigma+1}\,\theta}{\lambda(1-\sqrt{\theta})^{5/2}}, \tag{32}$$

$$|\mathrm{GPB}(n,\, \mathbf{p},\, F) - \exp\{\lambda(F-I)\}|_h \;\leq\; \frac{0.82\,\lfloor h+1\rfloor\,\theta}{\lambda^{3/2}(1-\sqrt{\theta})^{3}}, \tag{33}$$

where, for (32), *we assume that* F *has finite variance* σ^2.

Remark 2.6 (a) Inequality (30) is an improvement of formula (10) in Roos [30]. Further, in contrast to (1.9) in Čekanavičius [10], the bound (31) requires more restrictive assumptions but is, however, of a better order and contains an explicit constant. Inequalities (32) and (33) seem to be completely new.

(b) In (30), the constant $3/(2\mathrm{e})$ cannot be replaced by a smaller one. Indeed, a stronger result has been shown in Theorem 1 in [30]: we have

$$\lim_{r\downarrow 0}\Big(\sup \frac{1}{\theta}\|\mathrm{GPB}(n,\, \mathbf{p},\, F) - \exp\{\lambda(F-I)\}\|\Big) = \frac{3}{2\mathrm{e}},$$

where the sup is over all $n \in \mathbb{N}$, $p_1,\dots,p_n \in [0,1]$, $F \in \mathcal{F}$, such that $\theta = \lambda_2/\lambda \leq r$.

3 Auxiliary smoothness estimates

3.1 Some known estimates

The following lemma collects some basic norm estimates.

Lemma 3.1 *Let* $F \in \mathcal{F}$, $j, n \in \mathbb{Z}_+$, $p = 1-q \in (0,1)$, *and* $t \in (0,\infty)$. *Then*

$$\|(F-I)^2\exp\{t(F-I)\}\| \;\leq\; \frac{3}{t\mathrm{e}}, \tag{34}$$

$$\|(F-I)^j(I+p(F-I))^n\| \;\leq\; \binom{n+j}{j}^{-1/2}(pq)^{-j/2}, \tag{35}$$

If $j \in \mathbb{N}$, *then*

$$|(I_1-I)^j(I+p(I_1-I))^n|_0 \leq \frac{\sqrt{\mathrm{e}}}{2}\Big(1+\sqrt{\frac{\pi}{2j}}\Big)\Big(\frac{n}{n+j+1}\Big)^{(n+j+1)/2}\Big(\frac{j}{npq}\Big)^{(j+1)/2}. \tag{36}$$

If $n \in \mathbb{N}$ *and* $j \in \{1,\dots,n\}$, *then*

$$\frac{n^n}{j^j(n-j)^{n-j}} \leq \mathrm{e}\sqrt{j}\binom{n}{j}. \tag{37}$$

Inequality (34) was proved in Roos ([30], formula (29)). For the proof of (35)–(37), see Lemma 4, formula (35), and Lemma 3 in Roos [29], respectively. Note that (37) can be used to estimate the right-hand side of (35).

3.2 General symmetric distributions

The goal of this section is Proposition 3.1 below, for the proof of which we need the following two lemmas. The next lemma is also used in other proofs. Note that the first bound in (38) is required in the proof of Theorem 2.7, while the second one is used in Propositions 3.1 and 3.4.

Lemma 3.2 *Let $F \in \mathcal{F}$, $r \in [0,1)$, and, for $z \in \mathbb{C}$,*

$$g(z) = \frac{1}{(z-1)^2}\Big[\prod_{j=1}^{n}(1+p_j(z-1)) - \exp\{\lambda(z-1)\}\Big]\exp\{-r\lambda(z-1)\}.$$

If $\theta = \lambda_2/\lambda \leq 1-r$, then

$$\|g(F)\| \leq \min\Big\{\frac{\lambda_2}{2\sqrt{1-\theta/(1-r)}},\, 1.69\,\lambda_2\Big\}. \tag{38}$$

Proof. Let the so-called Charlier coefficients $a_m \in \mathbb{R}$, $(m \in \mathbb{Z}_+)$, be defined by

$$\mathrm{e}^{-\lambda z}\prod_{j=1}^{n}(1+p_j z) = 1 + \sum_{m=2}^{\infty} a_m z^m, \qquad (z \in \mathbb{C},\ a_0 = 1,\ a_1 = 0).$$

Using Shorgin's ([33], Theorem 1) recursive formula

$$a_m = \frac{1}{m}\sum_{k=0}^{m-2}(-1)^{m+k+1}a_k\lambda_{m-k}, \qquad (m \in \{2,3,\dots\}),$$

we derive

$$a_2 = -\frac{\lambda_2}{2}, \qquad a_3 = \frac{\lambda_3}{3}, \qquad a_4 = \frac{\lambda_2^2}{8} - \frac{\lambda_4}{4}, \qquad a_5 = \frac{\lambda_5}{5} - \frac{\lambda_2\lambda_3}{6},$$

and hence

$$|a_3| \leq \frac{\lambda_2^{3/2}}{3}, \quad |a_4| \leq \sum_{j=1}^{n} p_j^2\Big|\frac{p_j^2}{4} - \frac{\lambda_2}{8}\Big| \leq \frac{\lambda_2^2}{8}, \qquad |a_5| \leq \sum_{j=1}^{n} p_j^2\Big|\frac{p_j^3}{5} - \frac{\lambda_3}{6}\Big| \leq \frac{\lambda_2\lambda_3}{6} \leq \frac{\lambda_2^{5/2}}{6}.$$

As has been shown by Shorgin ([33], Lemma 5), $|a_m| \leq (\lambda_2\,\mathrm{e}/m)^{m/2}$ for $m \in \{2,3,\dots\}$. For $z \in \mathbb{C}$, $t \in \mathbb{R}$, and $m \in \mathbb{Z}_+$, let

$$\mathrm{Ch}(m;z,t) = \sum_{k=0}^{m}\binom{m}{k}\binom{z}{k}k!\,(-t)^{m-k}$$

be the Charlier polynomial of degree m with parameter t. It is well-known that, for $F \in \mathcal{F}$, $t \in \mathbb{R}$, and $m \in \mathbb{Z}_+$,

$$t^m(F-I)^m\exp\{t(F-I)\} = \sum_{k=0}^{\infty}\mathrm{e}^{-t}\frac{t^k}{k!}\,\mathrm{Ch}(m;k,t)F^k$$

and that the Charlier polynomials are orthogonal with respect to the Poisson counting density, i.e., for $t \in [0, \infty)$ and $j, m \in \mathbb{Z}_+$,

$$\sum_{k=0}^{\infty} \mathrm{e}^{-t} \frac{t^k}{k!} \operatorname{Ch}(j; k, t) \operatorname{Ch}(m; k, t) = \begin{cases} t^j\, j!, & \text{if } j = m, \\ 0, & \text{otherwise,} \end{cases}$$

see Chihara ([13], p. 3–4). Set $t = (1-r)\lambda$. Hence, for arbitrary $\ell \in \{2, 3, \dots\}$, we obtain

$$\begin{aligned} \|g(F)\| &= \Big\| \sum_{m=2}^{\infty} a_m (F-I)^{m-2} \exp\{t(F-I)\} \Big\| \\ &\le \Big\| \sum_{m=2}^{\ell} a_m (F-I)^{m-2} \exp\{t(F-I)\} \Big\| + \Big\| \sum_{m=\ell+1}^{\infty} a_m (F-I)^{m-2} \exp\{t(F-I)\} \Big\| \\ &= T_1 + T_2, \quad \text{say.} \end{aligned}$$

On the one hand, using Cauchy's inequality,

$$\begin{aligned} T_1 &= \Big\| \sum_{m=2}^{\ell} \frac{a_m}{t^{m-2}} \sum_{k=0}^{\infty} \mathrm{e}^{-t} \frac{t^k}{k!} \operatorname{Ch}(m-2; k, t) F^k \Big\| \le \sum_{k=0}^{\infty} \mathrm{e}^{-t} \frac{t^k}{k!} \Big| \sum_{m=2}^{\ell} \frac{a_m}{t^{m-2}} \operatorname{Ch}(m-2; k, t) \Big| \\ &\le \Big(\sum_{k=0}^{\infty} \mathrm{e}^{-t} \frac{t^k}{k!} \Big[\sum_{m=2}^{\ell} \frac{a_m}{t^{m-2}} \operatorname{Ch}(m-2; k, t) \Big]^2 \Big)^{1/2} \\ &= \Big(\sum_{m(1)=2}^{\ell} \sum_{m(2)=2}^{\ell} \frac{a_{m(1)}\, a_{m(2)}}{t^{m(1)+m(2)-4}} \sum_{k=0}^{\infty} \mathrm{e}^{-t} \frac{t^k}{k!} \operatorname{Ch}(m(1)-2; k, t) \operatorname{Ch}(m(2)-2; k, t) \Big)^{1/2} \\ &= \Big(\sum_{m=2}^{\ell} \frac{a_m^2}{t^{2m-4}}\, t^{m-2} (m-2)! \Big)^{1/2} \le \Big(\sum_{m=2}^{\infty} a_m^2 \frac{(m-2)!}{t^{m-2}} \Big)^{1/2}. \end{aligned}$$

On the other hand, we have

$$T_2 \le \sum_{m=\ell+1}^{\infty} |a_m| 2^{m-2} \le \sum_{m=\ell+1}^{\infty} \Big(\frac{\lambda_2\, \mathrm{e}}{m} \Big)^{m/2} 2^{m-2} \overset{(\ell \to \infty)}{\longrightarrow} 0.$$

Hence

$$\begin{aligned} \|g(F)\| &\le \Big(\sum_{m=2}^{\infty} a_m^2 \frac{(m-2)!}{t^{m-2}} \Big)^{1/2} \le \Big(\frac{\lambda_2^2}{4} + \frac{\lambda_2^3}{9\,t} + \frac{\lambda_2^4}{32\,t^2} + \frac{\lambda_2^5}{6\,t^3} + \sum_{m=6}^{\infty} \Big(\frac{\lambda_2\, \mathrm{e}}{m} \Big)^m \frac{(m-2)!}{t^{m-2}} \Big)^{1/2} \\ &= \frac{\lambda_2}{2} \Big(1 + \frac{4x}{9} + \frac{x^2}{8} + \frac{2x^3}{3} + \sum_{m=6}^{\infty} x^{m-2} \frac{4\, \mathrm{e}^m\, m!}{m^{m+1}(m-1)} \Big)^{1/2}, \end{aligned} \tag{39}$$

where $x = \lambda_2/t = \theta/(1-r)$. If, in (39), we let $x = 1$, we arrive at the second inequality in (38). Moreover, (39) together with Stirling's formula

$$m! = \sqrt{2\pi}\, m^{m+1/2} \exp\{-m + \vartheta_m\}, \qquad \vartheta_m \in \Big[\frac{1}{12m+1}, \frac{1}{12m} \Big],\ m \in \mathbb{N}, \tag{40}$$

(see Feller [17], p. 54) leads also to

$$\|g(F)\| \le \frac{\lambda_2}{2} \Big(1 + x + x^2 + x^3 + \sum_{m=6}^{\infty} x^{m-2} \frac{4\sqrt{2\pi}\, \mathrm{e}^{1/(12m)}}{m^{1/2}(m-1)} \Big)^{1/2} \le \frac{\lambda_2}{2\sqrt{1-x}},$$

which yields the first inequality in (38). □

Lemma 3.3 *Let $F \in \mathcal{S}$, $j \in \mathbb{Z}_+$, $t \in (0,\infty)$, and $h \in [0,\infty)$. Then*

$$
\begin{aligned}
|(F-I)^j \exp\{t(F-I)\}| &\leq \frac{C(j)}{t^j}, &(41)\\
|(F-I)^j \exp\{t(F-I)\}|_h &\leq \frac{C(j)}{t^j}\,\widetilde{Q}_h^{1/(2j+1)}(|\ln \widetilde{Q}_h|+1)^{6j(j+1)/(2j+1)}, &(42)
\end{aligned}
$$

where $\widetilde{Q}_h := \widetilde{Q}_{h,t,F} := |\exp\{4^{-1}t(F-I)\}|_h$.

For the proof, see Theorem 1.1 in Čekanavičius [7]. Further comments can be found in Čekanavičius and Roos ([12], Lemma 4.4).

Proposition 3.1 *Let $F \in \mathcal{S}$, $j \in \mathbb{Z}_+$, $n \in \mathbb{N}$, $p = 1-q \in (0,1)$, and $h \in [0,\infty)$. Then*

$$
\begin{aligned}
|(F-I)^j(I+p(F-I))^n| &\leq \frac{C(j)}{q\,(npq)^j}, &(43)\\
|(F-I)^j(I+p(F-I))^n|_h &\leq \frac{C(j)}{q\,(npq)^j}\,\widetilde{Q}_h^{1/(2j+3)}(|\ln \widetilde{Q}_h|+1)^{6(j+1)(j+2)/(2j+3)}, &(44)
\end{aligned}
$$

where $\widetilde{Q}_h := \widetilde{Q}_{h,npq,F} := |\exp\{4^{-1}\,npq\,(F-I)\}|_h$.

Proof. We use the second bound in (38) under the assumption that $p_1 = \cdots = p_n = p$ and $r = q$. Taking into account (41), we obtain

$$
\begin{aligned}
|(F-I)^j(I+p(F-I))^n| &\leq 2|(F-I)^{j+1}\exp\{npq(F-I)\}|\,\|g(F)\|\\
&\quad + |(F-I)^j\exp\{np(F-I)\}|\\
&\leq \frac{C(j)\,np^2}{(npq)^{j+1}} + \frac{C(j)}{(np)^j} \leq \frac{C(j)}{q\,(npq)^j},
\end{aligned}
$$

which implies (43). Inequality (44) can be shown in the same way with the help of (42). □

3.3 Symmetric distributions on the integers

In Čekanavičius and Roos ([12], Lemma 4.6), the following lemma was shown.

Lemma 3.4 *Let $j \in \mathbb{Z}_+$ and $t \in (0,\infty)$. If $F \in \mathcal{S}$ is concentrated on the set $\mathbb{Z}\setminus\{0\}$, then*

$$
\begin{aligned}
\|(F-I)^j \exp\{t(F-I)\}\| &\leq 3.6\, j^{1/4}\sqrt{1+\sigma}\Big(\frac{j}{t\mathrm{e}}\Big)^j, \qquad (j \neq 0), &(45)\\
|(F-I)^j \exp\{t(F-I)\}|_0 &\leq 2\Big(\frac{j+1/2}{t\mathrm{e}}\Big)^{j+1/2}, &(46)
\end{aligned}
$$

where, for (45)*, we assume that F has finite variance σ^2.*

In Proposition 3.2 below, we show similar bounds in the context of the compound binomial distribution, for the proof of which the following four lemmas are used. Some further notation is necessary. The Fourier transform of a finite signed measure $W \in \mathcal{M}$ is denoted by $\widehat{W}(x) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}W(y)$, $(x \in \mathbb{R})$. Here, i denotes the complex unit. It is easy to check that, for $V, W \in \mathcal{M}$ and $a, x \in \mathbb{R}$,

$$
\widehat{\exp\{W\}}(x) = \exp\{\widehat{W}(x)\}, \quad \widehat{VW}(x) = \widehat{V}(x)\widehat{W}(x), \quad \widehat{I_a}(x) = \mathrm{e}^{\mathrm{i}xa}, \quad \widehat{I}(x) = 1.
$$

Lemma 3.5 *Let $W \in \mathcal{M}$ be concentrated on $\mathbb{Z}$ satisfying $\sum_{k\in\mathbb{Z}} |k|\,|W(\{k\})| < \infty$. Then, for all $a \in \mathbb{R}$ and $b \in (0,\infty)$,*

$$\|W\|^2 \leq \frac{1+b\pi}{2\pi} \int_{-\pi}^{\pi} \Big(|\widehat{W}(x)|^2 + \frac{1}{b^2}\Big|\frac{\mathrm{d}}{\mathrm{d}x}\big(\mathrm{e}^{-\mathrm{i}xa}\widehat{W}(x)\big)\Big|^2\Big)\,\mathrm{d}x. \tag{47}$$

Further,

$$|W|_0 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{W}(x)|\,\mathrm{d}x. \tag{48}$$

For a proof of (47), see Presman ([27], Lemma on p. 419). Inequality (48) follows from the Fourier inversion formula; see also Čekanavičius and Roos ([12], Lemma 4.5). The following lemma is a version of the Kolmogorov–Rogozin inequality for concentration functions.

Lemma 3.6 *Let $n \in \mathbb{N}$, $F_1,\dots,F_n \in \mathcal{F}$ and $h \in [0,\infty)$. For $j \in \{1,\dots,n\}$, let F_j^{-1} denote the quantile function of F_j, that is, for $y \in (0,1)$, we have $F_j^{-1}(y) = \inf\{x \in \mathbb{R} : F_j((-\infty,x]) \geq y\}$. Let Y be a random variable, uniformly distributed on $(0,1/2)$, and set $\kappa = n^{-1}\sum_{j=1}^n \mathrm{P}(F_j^{-1}(1-Y) - F_j^{-1}(Y) \leq h)$. Then $\kappa \leq n^{-1}\sum_{j=1}^n |F_j|_h$ and*

$$\Big|\prod_{j=1}^n F_j\Big|_h \leq \Big(\frac{1-\kappa^{n+1}}{(n+1)(1-\kappa)}\Big)^{1/2}. \tag{49}$$

Proof. The assertion is a simple consequence of arguments by Kolmogorov [21], Rogozin ([28], Theorem 1, p. 95), and Le Cam ([25], Theorem 2, p. 411); cf. also Roos ([32], Proposition 5). Indeed, it can be shown that

$$\Big|\prod_{j=1}^n F_j\Big|_h \leq \mathrm{E}\frac{1}{\sqrt{Z+1}},$$

where $Z = \sum_{j=1}^n Z_j$ is the sum of independent Bernoulli random variables $Z_1,\dots,Z_n$ with success probabilities $p_j := \mathrm{P}(Z_j = 1) = \mathrm{P}(F_j^{-1}(1-Y) - F_j^{-1}(Y) > h)$, $(j \in \{1,\dots,n\})$; (see [25], p. 411). In order to estimate this mean, we proceed as Le Cam: using the integral representation of the Gamma function, we get, for $m \in \mathbb{Z}_+$,

$$\frac{1}{\sqrt{m+1}} = \frac{1}{\Gamma(1/2)}\int_0^\infty \frac{\mathrm{e}^{-x(m+1)}}{\sqrt{x}}\,\mathrm{d}x.$$

Hence, by Fubini's theorem,

$$\mathrm{E}\frac{1}{\sqrt{Z+1}} = \frac{1}{\Gamma(1/2)}\int_0^\infty \frac{\mathrm{e}^{-x}}{\sqrt{x}}\mathrm{E}(\mathrm{e}^{-xZ})\,\mathrm{d}x = \frac{1}{\Gamma(1/2)}\int_0^\infty \frac{\mathrm{e}^{-x}}{\sqrt{x}}\prod_{j=1}^n (1 + p_j(\mathrm{e}^{-x}-1))\,\mathrm{d}x.$$

Le Cam has now used the inequality $1 + x \leq \mathrm{e}^x$, $(x \in \mathbb{R})$. If we proceed with the better inequality between the arithmetic and geometric means, we obtain, together with the Jensen inequality,

$$\mathrm{E}\frac{1}{\sqrt{Z+1}} \leq \mathrm{E}\frac{1}{\sqrt{M+1}} \leq \Big(\mathrm{E}\frac{1}{M+1}\Big)^{1/2} = \Big(\frac{1-(1-\overline{p})^{n+1}}{(n+1)\overline{p}}\Big)^{1/2},$$

where M is a binomial distributed random variable with parameters n and $\overline{p} = n^{-1}\sum_{j=1}^n p_j$. The assertion is shown. □

Lemma 3.7 *Let $b \in (0, \infty)$, $p = 1 - q \in (0, 1)$, and let $F \in \mathcal{S}$ be concentrated on the set $\mathbb{Z} \setminus \{0\}$. Then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - p(1 - \widehat{F}(x))|^b \, \mathrm{d}x \le \sqrt{\frac{2}{bp}}. \tag{50}$$

Proof. Due to the Parseval equality, for each $G \in \mathcal{F}$ concentrated on $\mathbb{Z}$, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\widehat{G}(x)|^2 \, \mathrm{d}x = \sum_{k=-\infty}^{\infty} \big(G(\{k\})\big)^2 \le |G|_0.$$

Therefore, Lemma 3.6 can be applied to $G = I + p(F - I)$ with $\kappa = q$, giving

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - p(1 - \widehat{F}(x))|^b \, \mathrm{d}x &\le \frac{1}{2\pi} \int_{-\pi}^{\pi} |(1 - p(1 - \widehat{F}(x)))^{\lfloor b/2 \rfloor}|^2 \, \mathrm{d}x \\ &\le |G^{\lfloor b/2 \rfloor}|_0 \le \frac{1}{\sqrt{(\lfloor b/2 \rfloor + 1)p}} \le \sqrt{\frac{2}{bp}}. \end{aligned}$$

The lemma is proved. □

Lemma 3.8 *For $a, b \in (0, \infty)$ and $p = 1 - q \in (0, 1)$, we have*

$$\sup_{y \in [0,2]} y^a |1 - py|^b \le \Big(\frac{b}{a + b}\Big)^{a+b} \Big(\frac{a}{bpq}\Big)^a \le \Big(\frac{a}{\mathrm{e}bpq}\Big)^a. \tag{51}$$

Proof. For $y \in [0, \infty)$, let $f(y) = y^a |1 - py|^b$. It is easily seen that

$$\sup_{y \in [0,2]} f(y) \le \max\Big\{f(2),\, f\Big(\frac{a}{(a + b)p}\Big)\Big\} = \max\Big\{2^a |1 - 2p|^b,\, \frac{a^a\, b^b}{(a + b)^{a+b} p^a}\Big\}.$$

Using the simple fact that

$$\sup_{p \in [0,1]} |1 - 2p|^b (pq)^a \le \frac{a^a\, b^{b/2}}{2^a (2a + b)^{a+b/2}},$$

we obtain

$$\sup_{y \in [0,2]} f(y) \le \frac{a^a}{(pq)^a} \max\Big\{\frac{b^{b/2}}{(2a + b)^{a+b/2}},\, \frac{b^b}{(a + b)^{a+b}}\Big\} = \frac{a^a\, b^b}{(a + b)^{a+b} (pq)^a}.$$

The proof is easily completed. □

Proposition 3.2 *Let $j \in \mathbb{Z}_+$, $n \in \mathbb{N}$, and $p = 1 - q \in (0, 1)$. If $F \in \mathcal{S}$ is concentrated on the set $\mathbb{Z} \setminus \{0\}$, then*

$$\|(F - I)^j (I + p(F - I))^n\| \le 6.73 \sqrt{\sigma}\, \frac{j}{q^{1/4}} \Big(\frac{j}{\mathrm{e}\, npq}\Big)^j, \qquad (j \ne 0), \tag{52}$$

$$|(F - I)^j (I + p(F - I))^n|_0 \le 2\sqrt{\mathrm{e}q} \Big(\frac{j + 1/2}{\mathrm{e}\, npq}\Big)^{j+1/2}, \tag{53}$$

where, for (52), we assume that F has finite variance σ^2. If $F = 2^{-1}(I_{-1} + I_1)$, then

$$\|(F - I)^j (I + p(F - I))^n\| \le \frac{j!}{(pq)^j} \sqrt{\frac{n!}{(n + 2j)!}} \le \frac{j!}{((n + 1)pq)^j}. \tag{54}$$

Proof. Set $W = (F-I)^j(I+p(F-I))^n$. Let us first prove (54). Under the present assumptions, we have

$$F-I = -\frac{1}{2}(I_{-1}-I)(I_1-I).$$

Therefore, using Lemma 5 in Roos [31], we obtain that

$$\begin{aligned}\|(F-I)^j(I+p(F-I))^n\| &= \frac{1}{2^j}\Big\|(I_{-1}-I)^j(I_1-I)^j\Big(I+\frac{p}{2}(I_{-1}-I)+\frac{p}{2}(I_1-I)\Big)^n\Big\| \\ &\leq \frac{1}{2^j}\Big(\frac{(j!)^2\,n!}{(n+2j)!\,(pq/2)^{2j}}\Big)^{1/2},\end{aligned}$$

from which (54) follows. Now we prove (53). For $x\in\mathbb{R}$, we have

$$\begin{aligned}\widehat{F}(x) &= 2\sum_{k=1}^{\infty}F(\{k\})\cos(kx), \qquad 1-\widehat{F}(x) = 4\sum_{k=1}^{\infty}F(\{k\})\sin^2\Big(\frac{kx}{2}\Big)\in[0,2], \\ \widehat{W}(x) &= (\widehat{F}(x)-1)^j(1+p(\widehat{F}(x)-1))^n.\end{aligned}$$

Let $n_1,n_2\in(0,\infty)$ such that $n=n_1+n_2$. From (48), (50), and (51), it follows

$$\begin{aligned}|W|_0 &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\widehat{F}(x))^j|1-p(1-\widehat{F}(x))|^n\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\widehat{F}(x))^j|1-p(1-\widehat{F}(x))|^{n_1}|1-p(1-\widehat{F}(x))|^{n_2}\,\mathrm{d}x \\ &\leq \Big(\sup_{y\in[0,2]}y^j|1-py|^{n_1}\Big)\frac{1}{2\pi}\int_{-\pi}^{\pi}|1-p(1-\widehat{F}(x))|^{n_2}\,\mathrm{d}x \\ &\leq \sqrt{\frac{2}{n_2p}}\Big(\frac{j}{\mathrm{e}n_1pq}\Big)^j. \qquad (55)\end{aligned}$$

It is easily seen that the upper bound in (55) attains its minimum for $n_1 = jn/(j+1/2)$. This leads us to (53). Now we show (52) by means of (47) with $a=0$. The value of $b\in(0,\infty)$ will be chosen later. We assume that $j\in\mathbb{N}$. First note that, from the above, it follows that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|\widehat{W}(x)|^2\,\mathrm{d}x &\leq 2\sqrt{\mathrm{e}q}\Big(\frac{2j+1/2}{2\mathrm{e}\,npq}\Big)^{2j+1/2} = 2\sqrt{\mathrm{e}q}\Big(\frac{4j+1}{4j}\Big)^{2j+1/2}\Big(\frac{j}{\mathrm{e}\,npq}\Big)^{2j+1/2} \\ &\leq 2\sqrt{\mathrm{e}}\Big(\frac{5}{4}\Big)^{5/2}\sqrt{q}\Big(\frac{j}{\mathrm{e}\,npq}\Big)^{2j+1/2}. \qquad (56)\end{aligned}$$

We introduce an arbitrary quantity $t\in(1,\infty)$. Let us first assume that

$$t\leq 4npq. \qquad (57)$$

Then, we have $t\leq n$ and $t\leq 4np$. Using Cauchy's inequality, we get

$$\Big|\frac{\mathrm{d}}{\mathrm{d}x}\widehat{F}(x)\Big| = 4\Big|\sum_{k=1}^{\infty}kF(\{k\})\sin\Big(\frac{kx}{2}\Big)\cos\Big(\frac{kx}{2}\Big)\Big| \leq \sqrt{2}\sigma(1-\widehat{F}(x))^{1/2}, \qquad (x\in\mathbb{R})$$

and therefore, for $x\in\mathbb{R}$, $y=1-\widehat{F}(x)$, $k_1 = 2(n-1)-k_2$, $k_2 = 2cj^{-1}(n-1)$, and arbitrary $c\in(0,1)$, we derive

$$\begin{aligned}\Big|\frac{\mathrm{d}}{\mathrm{d}x}\widehat{W}(x)\Big|^2 &\leq \Big|\frac{\mathrm{d}}{\mathrm{d}x}\widehat{F}(x)\Big|^2(1-\widehat{F}(x))^{2j-2}|j-(j+n)p(1-\widehat{F}(x))|^2|1-p(1-\widehat{F}(x))|^{2(n-1)} \\ &\leq 2\,\sigma^2\,y^{2j-1}(j-(j+n)py)^2|1-py|^{k_1}|1-p(1-\widehat{F}(x))|^{k_2}. \qquad (58)\end{aligned}$$

On the one hand, if $j \le (j+n)py$, then (58) together with (51) and (57) gives

$$\begin{aligned}
\Big|\frac{\mathrm{d}}{\mathrm{d}x}\widehat{W}(x)\Big|^2 &\le 2\,\sigma^2(j+n)^2p^2 \sup_{\widetilde{y}\in[0,2]}\Big(\widetilde{y}^{2j+1}|1-p\widetilde{y}|^{k_1}\Big)|1-p(1-\widehat{F}(x))|^{k_2}\\
&\le 2\,\sigma^2(j+n)^2p^2\Big(\frac{2j+1}{\mathrm{e}k_1pq}\Big)^{2j+1}|1-p(1-\widehat{F}(x))|^{k_2}\\
&= \frac{2\sigma^2}{\mathrm{e}q}\,j^3\,np\,\Big(\frac{j+n}{jn}\Big)^2\Big(\frac{(j+1/2)n}{(j-c)(n-1)}\Big)^{2j+1}\Big(\frac{j}{\mathrm{e}\,npq}\Big)^{2j}|1-p(1-\widehat{F}(x))|^{k_2}\\
&\le \frac{2\sigma^2}{q}\,j^3\sqrt{(n-1)n}\,p\,h_1(c,j,t)\Big(\frac{j}{\mathrm{e}\,npq}\Big)^{2j}|1-p(1-\widehat{F}(x))|^{k_2}, \qquad (59)
\end{aligned}$$

where

$$h_1(c,j,t) = \frac{1}{\mathrm{e}}\Big(\frac{j+t}{jt}\Big)^2\Big(\frac{(j+1/2)t}{(j-c)(t-1)}\Big)^{2j+1}\sqrt{\frac{t}{t-1}}.$$

On the other hand, if $j \ge (j+n)py$, then (58) together with (51) and (57) yields

$$\begin{aligned}
\Big|\frac{\mathrm{d}}{\mathrm{d}x}\widehat{W}(x)\Big|^2 &\le 2\,\sigma^2 j^2 \sup_{\widetilde{y}\in[0,2]}\Big(\widetilde{y}^{2j-1}|1-p\widetilde{y}|^{k_1}\Big)|1-p(1-\widehat{F}(x))|^{k_2}\\
&\le 2\,\sigma^2 j^2\Big(\frac{2j-1}{\mathrm{e}k_1pq}\Big)^{2j-1}|1-p(1-\widehat{F}(x))|^{k_2}\\
&= 2\sigma^2\,j\,\mathrm{e}\,npq\,\Big(\frac{(j-1/2)n}{(j-c)(n-1)}\Big)^{2j-1}\Big(\frac{j}{\mathrm{e}\,npq}\Big)^{2j}|1-p(1-\widehat{F}(x))|^{k_2}\\
&\le \frac{2\sigma^2}{q}\,j^3\sqrt{(n-1)n}\,p\,h_2(c,j,t)\Big(\frac{j}{\mathrm{e}\,npq}\Big)^{2j}|1-p(1-\widehat{F}(x))|^{k_2}, \qquad (60)
\end{aligned}$$

where

$$h_2(c,j,t) = \frac{\mathrm{e}}{j^2}\Big(\frac{(j-1/2)t}{(j-c)(t-1)}\Big)^{2j-1}\sqrt{\frac{t}{t-1}}.$$

Combining (59) and (60), we see that, under the assumption of (57), we have

$$\Big|\frac{\mathrm{d}}{\mathrm{d}x}\widehat{W}(x)\Big|^2 \le \frac{2\sigma^2}{q}\,j^3\sqrt{(n-1)n}\,p\,h_3(c,j,t)\Big(\frac{j}{\mathrm{e}\,npq}\Big)^{2j}|1-p(1-\widehat{F}(x))|^{k_2}, \qquad (61)$$

where

$$h_3(c,j,t) = \max\{h_1(c,j,t),\, h_2(c,j,t)\}.$$

Using (61) and (50), we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\frac{\mathrm{d}}{\mathrm{d}x}\widehat{W}(x)\Big|^2\,\mathrm{d}x \le \frac{2\sigma^2\,j^{7/2}\sqrt{np}}{q\sqrt{c}}\,h_3(c,j,t)\Big(\frac{j}{\mathrm{e}\,npq}\Big)^{2j}. \qquad (62)$$

From (47) in combination with (56) and (62), we conclude that, for arbitrary $b \in (0,\infty)$,

$$\begin{aligned}
\|W\|^2 &\le (1+b\pi)\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\widehat{W}(x)|^2\,\mathrm{d}x + \frac{1}{2\pi\,b^2}\int_{-\pi}^{\pi}\Big|\frac{\mathrm{d}}{\mathrm{d}x}\widehat{W}(x)\Big|^2\,\mathrm{d}x\Big)\\
&\le \Big(\frac{j}{\mathrm{e}\,npq}\Big)^{2j}(1+b\pi)\Big[2\Big(\frac{5}{4}\Big)^{5/2}\sqrt{\frac{j}{np}} + \frac{2\sigma^2\,j^{7/2}\sqrt{np}}{b^2q\sqrt{c}}\,h_3(c,j,t)\Big]. \qquad (63)
\end{aligned}$$

Let $b_0 \in (0,\infty)$ be arbitrary and set $b = b_0\sigma j^{3/2}\sqrt{np/q}$. Then (63) implies that

$$\begin{aligned}\|W\|^2 &\leq \Big(\frac{j}{\mathrm{e}\,npq}\Big)^{2j}\Big(1+b_0\sigma j^{3/2}\sqrt{\frac{np}{q}}\pi\Big)\sqrt{\frac{j}{np}}\Big[2\Big(\frac{5}{4}\Big)^{5/2}+\frac{2\,h_3(c,j,t)}{b_0^2\sqrt{c}}\Big]\\ &\leq \frac{\sigma}{\sqrt{q}}j^2\Big(\frac{j}{\mathrm{e}\,npq}\Big)^{2j}T_1, \qquad (64)\end{aligned}$$

where

$$T_1 = T_1(b_0,c,j,t) = \Big(\sqrt{\frac{4}{t}}+b_0\pi\Big)\Big[2\Big(\frac{5}{4}\Big)^{5/2}+\frac{2\,h_3(c,j,t)}{b_0^2\sqrt{c}}\Big].$$

Let us now assume that, in contrast to (57),

$$4npq < t.$$

Under this assumption, we get from (35) and (37) that

$$\|W\|^2 \leq \mathrm{e}\sqrt{j}\Big(\frac{n}{n+j}\Big)^{n+j}\Big(\frac{j}{npq}\Big)^j \leq \mathrm{e}\sqrt{j}\frac{\sigma}{\sqrt{q}}\Big(\frac{j}{\mathrm{e}\,npq}\Big)^j\Big(\frac{t}{4\,npq}\Big)^j = \frac{\sigma}{\sqrt{q}}j^2\Big(\frac{j}{\mathrm{e}\,npq}\Big)^{2j}T_2, \qquad (65)$$

where

$$T_2 = T_2(j,t) = \frac{\mathrm{e}}{j^{3/2}}\Big(\frac{\mathrm{e}t}{4j}\Big)^j.$$

From (64) and (65), it now follows that, for each choice of $j,n\in\mathbb{N}$, $p = 1-q\in(0,1)$, $b_0\in(0,\infty)$, $c\in(0,1)$, and $t\in(1,\infty)$, we have

$$\|W\|^2 = \max\{T_1,T_2\}\frac{\sigma}{\sqrt{q}}j^2\Big(\frac{j}{\mathrm{e}\,npq}\Big)^{2j}.$$

It remains to show that $\max\{T_1,T_2\}\leq 45.22\leq(6.73)^2$. For given value of j, choose b_0, c, and t from the following table. It is easily verified that $\max\{T_1,\,T_2\}$ is then bounded by the values given in the last column.

j	b_0	c	t	$\max\{T_1,\,T_2\}\leq$
1	2.0525	0.14286	24.477	45.22
2	1.1921	0.18182	15.469	26.56
3	0.9071	0.20000	14.969	20.40
4	0.7529	0.21052	15.667	17.06
$5\leq j$	0.7529	0.21052	$4j/\mathrm{e}$	35.18

The lemma is shown. □

3.4 Centered distributions

The main result of this section is Proposition 3.3 below, the proof of which requires some preparation. Recall that, for $W\in\mathcal{M}$, we denote by $W = W^+ - W^-$ the Hahn–Jordan decomposition of W. We shall use the following property of the compound measures

$$\sup_{F\in\mathcal{F}}\|(F-I)^j(I+p(F-I))^n\| = \|(I_1-I)^j(I+p(I_1-I))^n\|, \qquad (66)$$

where $j,n\in\mathbb{Z}_+$ and $p=1-q\in[0,1]$. In what follows, we need the following smoothing estimate, which is proved by only a slight modification of inequalities of Esseen [16], Le Cam [24], and Ibragimov and Presman [20]. However, for the sake of completeness, we provide a proof.

Lemma 3.9 *Let $W_1, W_2 \in \mathcal{M}$ with $W_1(\mathbb{R}) = 0$ and set $W = W_1 + W_2$. For $y \in [0, \infty)$, let*

$$\rho(y) = \min\bigl\{|W^+|_{y-},\, |W^-|_{y-}\bigr\}.$$

Then, for arbitrary $\vartheta \in (0, \infty)$ and $r \in (0, 1)$,

$$|W| \leq \frac{1}{2r}\,\|W_1\| + \frac{1}{2\pi\, r}\int_{|t|<1/\vartheta} \Bigl|\frac{\widehat{W_2}(t)}{t}\Bigr|\,\mathrm{d}t + \frac{1+r}{2r}\,\rho(4\,\eta(r)\vartheta),$$

where $\eta(r) \in (0, \infty)$ is defined by the equation

$$\frac{1+r}{2} = \frac{2}{\pi}\int_0^{\eta(r)} \frac{\sin^2(x)}{x^2}\,\mathrm{d}x. \tag{67}$$

Proof. Le Cam ([24], Proposition 3, p. 182) has shown that, if $W \in \mathcal{M}$, $F \in \mathcal{F}$, $y \in [0, \infty)$, $h \in [0, |F|_y]$, and ρ is defined as above, then

$$(2h-1)|W| \leq |WF| + h\,\rho(y). \tag{68}$$

Let $\vartheta \in (0, \infty)$ and $F = F_\vartheta \in \mathcal{F}$ have the Lebesgue density $f_\vartheta(x) = 2\vartheta(\pi x^2)^{-1}\,\sin^2(x/(2\vartheta))$, $(x \in \mathbb{R})$ and characteristic function

$$\widehat{F_\vartheta}(t) = \begin{cases} 1 - \vartheta|t|, & \text{if } |t| < 1/\vartheta, \\ 0, & \text{if } |t| \geq 1/\vartheta, \end{cases}$$

for $t \in \mathbb{R}$. If $r \in (0, 1)$, $y = 4\eta(r)\vartheta$, and $h = (1+r)/2$, then we have

$$|F_\vartheta|_y \geq \int_{-y/2}^{y/2} f_\vartheta(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\eta(r)} \frac{\sin^2(x)}{x^2}\,\mathrm{d}x = \frac{1+r}{2} = h.$$

Therefore, (68) gives

$$r|W| \leq \frac{1}{2}\,\|W_1\| + |W_2 F_\vartheta| + \frac{1+r}{2}\,\rho(4\eta(r)\vartheta).$$

From the Fourier inversion formula and the Riemann–Lebesgue Lemma, it follows that, for $x \in \mathbb{R}$,

$$W_2F_\vartheta((-\infty, x]) = \frac{1}{2\pi}\int_{|t|<1/\vartheta} \mathrm{e}^{-\mathrm{i}tx}\,\frac{\widehat{W_2}(t)\widehat{F_\vartheta}(t)}{-\mathrm{i}t}\,\mathrm{d}t.$$

The proof is easily completed . □

Lemma 3.10 *For $F \in \mathcal{F}$, $W \in \mathcal{M}$ with $W(\mathbb{R}) = 0$, and $\vartheta \in (0, \infty)$, we have*

$$|(WF)^+|_{\vartheta-} \leq \frac{1}{2}\,\|W\|\,|F|_{\vartheta-}. \tag{69}$$

Proof. In view of the simple fact that $(WF)^+ \leq W^+F$, we obtain

$$|(WF)^+|_{\vartheta-} \leq \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} F((x, x+\vartheta] - y)\,\mathrm{d}W^+(y) \leq W^+(\mathbb{R})\,|F|_{\vartheta-} = \frac{1}{2}\,\|W\|\,|F|_{\vartheta-}.$$

The proof is completed. □

Lemma 3.11 *Let $A, B \in \mathcal{F}$, $\alpha = 1 - \beta \in (0,1]$, $G = \alpha A + \beta B$, $j, n \in \mathbb{Z}_+$, and $p = 1 - q \in (0,1)$. Then*

$$\|(A - I)^j (I + p(G - I))^n\| \le \binom{n+j}{j}^{-1/2} \Big(\frac{1 - p\beta}{pq\alpha}\Big)^{j/2}. \tag{70}$$

Proof. Set $a = 1 - b = q/(1 - p\beta)$. Then $b = p\alpha/(1 - p\beta)$ and, taking into account (35) and the Jensen inequality, we get

$$\begin{aligned}
&\|(A - I)^j (I + p(G - I))^n\| \\
&= \Big\|(A - I)^j (p\beta B + (1 - p\beta)(aI + bA))^n\Big\| \\
&\le \sum_{m=0}^{n} \binom{n}{m} (p\beta)^{n-m} (1 - p\beta)^m \|(A - I)^j (I + b(A - I))^m\| \\
&\le \frac{1}{(ab)^{j/2}} \sum_{m=0}^{n} \binom{n}{m} (p\beta)^{n-m} (1 - p\beta)^m \binom{m+j}{j}^{-1/2} \\
&\le \frac{1}{(ab)^{j/2}} \Big(\sum_{m=0}^{n} \binom{n}{m} (p\beta)^{n-m} (1 - p\beta)^m \binom{m+j}{j}^{-1}\Big)^{1/2} \\
&= \binom{n+j}{j}^{-1/2} \Big(\frac{1 - p\beta}{pq\alpha}\Big)^{j/2} \Big(\sum_{m=0}^{n} \binom{n+j}{m+j} (p\beta)^{n-m} (1 - p\beta)^{m+j}\Big)^{1/2},
\end{aligned}$$

from which the assertion follows. □

Proposition 3.3 *Let $F \in \mathcal{F}$, $j, n \in \mathbb{N}$, and $p = 1 - q \in (0,1)$. Then*

$$\inf_{u \in \mathbb{R}} |(I_u F - I)^j (I + p(I_u F - I))^n| \le 17.6 \Big(\frac{j}{\mathrm{e}}\Big)^{(j+1)/2} \frac{j}{(npq)^{j/2 + j/(2j+2)}}. \tag{71}$$

Proof. Let $W = (I_u F - I)^j (I + p(I_u F - I))^n$ for a given $u \in \mathbb{R}$. Let $w \in (0, \infty)$ be arbitrary. We first assume that

$$npq > w^{j+1}. \tag{72}$$

As in Le Cam ([24], Section 5) or Ibragimov and Presman ([20], p. 719), we can choose $u \in \mathbb{R}$, $\vartheta_-, \vartheta_+ \in [0, \infty)$, and $A, B \in \mathcal{F}$ such that

$$I_u F = \alpha A + \beta B, \qquad \alpha = 1 - \beta = \frac{w}{(npq)^{1/(j+1)}} \in (0,1), \tag{73}$$

$$(I_u F)((-\infty, -\vartheta_-]) \ge \frac{\alpha}{2}, \qquad (I_u F)([\vartheta_+, \infty)) \ge \frac{\alpha}{2}, \tag{74}$$

$$A((-\vartheta_-, \vartheta_+)) = 0, \qquad B([-\vartheta_-, \vartheta_+]) = 1, \qquad \int_{\mathbb{R}} x \,\mathrm{d}B(x) = 0. \tag{75}$$

Set $\vartheta = \max(\vartheta_-, \vartheta_+)$ and $\sigma^2 = \int_{\mathbb{R}} x^2 \,\mathrm{d}B(x)$. Then, as shown in Arak and Zaĭtsev ([1], Theorem 1.1.10, p. 16–17), for $t \in \mathbb{R}$, we have

$$|\widehat{B}(t) - 1| \le \frac{\sigma^2 t^2}{2}, \tag{76}$$

and, if $\vartheta > 0$ and $|t| \le 1/\vartheta$, then

$$\sigma^2 t^2 \le 1, \qquad \mathrm{Re}(\widehat{B}(t)) - 1 \le -\frac{\sigma^2 t^2}{3}. \tag{77}$$

Set

$$W_1 = \alpha^j (A-I)^j (I + p(I_u F - I))^n, \qquad W_2 = W - W_1.$$

Let us first derive a bound for $\|W_1\|$. Taking into account (70) and (37), we get

$$\begin{aligned} \|W_1\| &= \alpha^j \Big\| (A-I)^j (I + p(\alpha A + \beta B - I))^n \Big\| \\ &\leq \sqrt{\mathrm{e}}\, j^{1/4} \Big(\frac{n}{n+j}\Big)^{(n+j)/2} \Big(\frac{(q+p\alpha)\alpha j}{npq}\Big)^{j/2} \\ &\leq \frac{\mathrm{e}\, w^{j/2}}{j^{5/4}} \Big(\frac{j}{\mathrm{e}}\Big)^{(j+1)/2} \frac{j}{(npq)^{j/2+j/(2j+2)}}. \end{aligned} \tag{78}$$

If $\vartheta = 0$, then $B = I$ and $W = W_1$, and therefore

$$|W| \leq \frac{1}{2}\|W_1\| \leq T_1 \Big(\frac{j}{\mathrm{e}}\Big)^{(j+1)/2} \frac{j}{(npq)^{j/2+j/(2j+2)}},$$

where

$$T_1 := T_1(j,w) := \frac{\mathrm{e}\, w^{j/2}}{2\, j^{5/4}}. \tag{79}$$

Let us now assume that $\vartheta > 0$. Here, we shall apply Lemma 3.9, giving

$$|W| \leq \frac{1}{2r}\|W_1\| + \frac{1}{2\pi r}\int_{|t|<1/\vartheta} \Big|\frac{\widehat{W_2}(t)}{t}\Big|\,\mathrm{d}t + \frac{1+r}{2r}\lceil 4\,\eta(r)\rceil |W^+|_{\vartheta-}, \tag{80}$$

where $r \in (0,1)$ will be chosen later and $\eta(r)$ is defined by (67). Let

$$n_1 = n - \Big\lfloor \frac{n}{j+1} \Big\rfloor, \qquad n_2 = \Big\lfloor \frac{n}{j+1} \Big\rfloor.$$

We then have $n_1 \in \mathbb{N}$, $n_2 \in \mathbb{Z}_+$, and

$$\frac{1}{n_1} \leq \frac{j+1}{j\,n}, \qquad \frac{1}{n_2+1} \leq \frac{j+1}{n}. \tag{81}$$

Taking into account (69), (66), (35), (49), (37), (73)–(75), and (81), we obtain

$$\begin{aligned} |W^+|_{\vartheta-} &\leq \frac{1}{2}\,\|(I_1 - I)^j (I + p(I_1 - I))^{n_1}\|\,|(I + p(I_u F - I))^{n_2}|_{\vartheta-} \\ &\leq \frac{1}{2}\binom{n_1+j}{j}^{-1/2} (pq)^{-j/2} \frac{1}{\sqrt{(n_2+1)(1-q-p|I_u F|_{\vartheta-})}} \\ &\leq \sqrt{\frac{\mathrm{e}}{2}}\, j^{1/4} \Big(\frac{n_1}{n_1+j}\Big)^{(n_1+j)/2} \Big(\frac{j}{n_1 pq}\Big)^{j/2} \frac{1}{\sqrt{(n_2+1)p\alpha}} \\ &\leq \frac{\mathrm{e}}{j^{3/4}} \Big(\frac{j+1}{j}\Big)^{(j+1)/2} \sqrt{\frac{q}{2w}} \Big(\frac{j}{\mathrm{e}}\Big)^{(j+1)/2} \frac{j}{(npq)^{j/2+j/(2j+2)}}. \end{aligned} \tag{82}$$

It remains to estimate the integral in (80). For $t \in \mathbb{R}$, we have

$$\begin{aligned} \widehat{W_2}(t) &= [(\alpha(\widehat{A}(t)-1) + \beta(\widehat{B}(t)-1))^j - (\alpha(\widehat{A}(t)-1))^j][1 + p(\mathrm{e}^{\mathrm{i}tu}\widehat{F}(t) - 1)]^n \\ &= j\beta[\widehat{B}(t)-1]\int_0^1 [\alpha(\widehat{A}(t)-1) + x\beta(\widehat{B}(t)-1)]^{j-1}\mathrm{d}x\,[1 + p(\mathrm{e}^{\mathrm{i}tu}\widehat{F}(t)-1)]^n. \end{aligned} \tag{83}$$

It is easy to check that, for $t \in \mathbb{R}$,

$$|1 + p(\mathrm{e}^{\mathrm{i}tu}\widehat{F}(t) - 1)|^n \leq \exp\{-npq\,\alpha[1 - \mathrm{Re}(\widehat{A}(t))] - npq\,\beta[1 - \mathrm{Re}(\widehat{B}(t))]\}. \tag{84}$$

Let

$$\widetilde{n}_1 = \frac{(j-1)n}{j+1}, \qquad \widetilde{n}_2 = \frac{2n}{j+1}.$$

Note that, here, we do not need to assume $\widetilde{n}_1$ and $\widetilde{n}_2$ to be integers. Then, by using (83) and (84), for $t \in \mathbb{R}$, we get

$$\begin{aligned} |\widehat{W_2}(t)| \leq\ & j\beta|\widehat{B}(t) - 1| \exp\{-\widetilde{n}_2 pq\,\beta[1 - \mathrm{Re}(\widehat{B}(t))]\} \\ & \times \int_0^1 (\alpha + x\beta)^{j-1} \Big|1 - \Big(\frac{\alpha}{\alpha + x\beta}\widehat{A}(t) + \frac{x\beta}{\alpha + x\beta}\widehat{B}(t)\Big)\Big|^{j-1} \mathrm{d}x \\ & \times \exp\{-npq\,\alpha[1 - \mathrm{Re}(\widehat{A}(t))] - \widetilde{n}_1 pq\,\beta[1 - \mathrm{Re}(\widehat{B}(t))]\}. \end{aligned}$$

Since, for $G \in \mathcal{F}$ and $t \in \mathbb{R}$,

$$|1 - \widehat{G}(t)|^2 \leq 2[1 - \mathrm{Re}(\widehat{G}(t))], \tag{85}$$

and, for all $x, \zeta, k \in (0, \infty)$, $x^k \mathrm{e}^{-\zeta x} \leq \big(k/(\mathrm{e}\zeta)\big)^k$, we obtain, for $x \in (0,1)$ and $t \in \mathbb{R}$,

$$\begin{aligned} & (\alpha + x\beta)^{j-1} \Big|1 - \Big(\frac{\alpha}{\alpha + x\beta}\widehat{A}(t) + \frac{x\beta}{\alpha + x\beta}\widehat{B}(t)\Big)\Big|^{j-1} \\ & \qquad \times \exp\{-npq\,\alpha[1 - \mathrm{Re}(\widehat{A}(t))] - \widetilde{n}_1 pq\,\beta[1 - \mathrm{Re}(\widehat{B}(t))]\} \\ & \leq\ (2(\alpha + x\beta))^{(j-1)/2} \big[\alpha[1 - \mathrm{Re}(\widehat{A}(t))] + x\beta[1 - \mathrm{Re}(\widehat{B}(t))]\big]^{(j-1)/2} \\ & \qquad \times \exp\{-\widetilde{n}_1 pq\big(\alpha[1 - \mathrm{Re}(\widehat{A}(t))] + x\beta[1 - \mathrm{Re}(\widehat{B}(t))]\big)\} \\ & \leq\ \Big(\frac{j-1}{\mathrm{e}\widetilde{n}_1 pq}\Big)^{(j-1)/2}. \end{aligned}$$

This yields

$$|\widehat{W_2}(t)| \leq j\beta\Big(\frac{j-1}{\mathrm{e}\widetilde{n}_1 pq}\Big)^{(j-1)/2} |\widehat{B}(t) - 1| \exp\{-\widetilde{n}_2 pq\,\beta[1 - \mathrm{Re}(\widehat{B}(t))]\}. \tag{86}$$

Due to (76) and (77), we have

$$\begin{aligned} & \int_{|t|<1/\vartheta} \Big|\frac{\widehat{B}(t) - 1}{t}\Big| \exp\{-\widetilde{n}_2 pq\,\beta[1 - \mathrm{Re}(\widehat{B}(t))]\}\,\mathrm{d}t \\ & \qquad \leq\ \sigma^2 \int_0^\infty t \exp\Big\{-\frac{1}{3}\widetilde{n}_2 pq\,\beta(\sigma t)^2\Big\}\,\mathrm{d}t = \frac{3}{2\widetilde{n}_2 pq\beta}. \end{aligned} \tag{87}$$

From (86) and (87), it follows that

$$\int_{|t|<1/\vartheta} \Big|\frac{\widehat{W_2}(t)}{t}\Big|\,\mathrm{d}t \leq \frac{3\mathrm{e}}{4\sqrt{w}}\Big(\frac{j+1}{\mathrm{e}}\Big)^{(j+1)/2} \frac{j}{(npq)^{j/2 + j/(2j+2)}}. \tag{88}$$

Collecting the estimates (88), (82), (80), and (78), we get, under the assumption of (72) for all possible values of $\vartheta \geq 0$, that

$$|W| \leq T_2 \left(\frac{j}{\mathrm{e}}\right)^{(j+1)/2} \frac{j}{(npq)^{j/2 + j/(2j+2)}}, \tag{89}$$

where

$$\begin{aligned} T_2 &= T_2(j,r,w) \\ &= \frac{\mathrm{e}\,w^{j/2}}{2r\,j^{5/4}} + \frac{3\mathrm{e}}{8\pi r\sqrt{w}}\Big(\frac{j+1}{j}\Big)^{(j+1)/2} + \frac{1+r}{2r}\,\lceil 4\,\eta(r)\rceil\,\frac{\mathrm{e}}{j^{3/4}}\Big(\frac{j+1}{j}\Big)^{(j+1)/2}\frac{1}{\sqrt{2w}}. \end{aligned}$$

Here, we used the fact that $T_1 \leq T_2$. If, in contrast to (72), $npq \leq w^{j+1}$, then, by (66), (35), and (37), we have

$$\begin{aligned} |W| &\leq \frac{1}{2}\,\|W\| \leq \frac{\sqrt{\mathrm{e}}}{2}\,j^{1/4}\Big(\frac{n}{n+j}\Big)^{(n+j)/2}\Big(\frac{j}{npq}\Big)^{j/2}\frac{w^{j/2}}{(npq)^{j/(2j+2)}} \\ &\leq T_1\Big(\frac{j}{\mathrm{e}}\Big)^{(j+1)/2}\frac{j}{(npq)^{j/2+j/(2j+2)}}, \end{aligned} \tag{90}$$

where T_1 is defined in (79). Combining (89) and (90), we see that (89) is generally valid. Let $r = 0.6295$, giving $\lceil 4\eta(r)\rceil = 7$. If $j = 1$, then set $w = 16.6$; if $j = 2$, then set $w = 5$; and if $j \geq 3$, then set $w = 1$. It is not difficult to show that, under these assumptions, $T_2 \leq 17.6$. This proves the validity of (71). □

Corollary 3.1 *Let $F \in \mathcal{F}$, $j \in \mathbb{N}$, $t \in (0,\infty)$. Then*

$$\inf_{u\in\mathbb{R}} |(I_uF - I)^j \exp\{t(I_uF - I)\}| \leq 17.6\Big(\frac{j}{\mathrm{e}}\Big)^{(j+1)/2}\frac{j}{t^{j/2+j/(2j+2)}}. \tag{91}$$

Proof. Choose $n \in \mathbb{N}$ such that $t/n < 1$. By (71), there exists $u = u(n) \in \mathbb{R}$ such that

$$\Big|(I_uF - I)^j\Big(I + \frac{t}{n}(I_uF - I)\Big)^n\Big| \leq \frac{t}{n} + 17.6\Big(\frac{j}{\mathrm{e}}\Big)^{(j+1)/2}\frac{j}{(t(1-t/n))^{j/2+j/(2j+2)}}.$$

Then, in view of (30),

$$\begin{aligned} |(I_uF - I)^j \exp\{t(I_uF - I)\}| &\leq \Big|(I_uF - I)^j\Big(\exp\{t(I_uF - I)\} - \Big(I + \frac{t}{n}(I_uF - I)\Big)^n\Big)\Big| \\ &\quad + \Big|(I_uF - I)^j\Big(I + \frac{t}{n}(I_uF - I)\Big)^n\Big| \\ &\leq 2^{j-1}\Big\|\exp\{t(I_1 - I)\} - \Big(I + \frac{t}{n}(I_1 - I)\Big)^n\Big\| \\ &\quad + \frac{t}{n} + 17.6\Big(\frac{j}{\mathrm{e}}\Big)^{(j+1)/2}\frac{j}{(t(1-t/n))^{j/2+j/(2j+2)}} \\ &\to 17.6\Big(\frac{j}{\mathrm{e}}\Big)^{(j+1)/2}\frac{j}{t^{j/2+j/(2j+2)}}, \end{aligned}$$

as $n \to \infty$. This proves the assertion. □

It should be noted that Corollary 3.1 is an improvement of Theorem 3.1 in Čekanavičius [7] in the sense that it contains an explicit estimate.

3.5 Asymptotically sharp norm estimates

The next lemma is needed in the proof of Proposition 3.4 below. A proof can be found in Čekanavičius and Roos ([12], Lemma 4.7). Recall that φ_j, $(j \in \mathbb{Z}_+)$ is defined in (4).

Lemma 3.12 *Let $j \in \mathbb{Z}_+$, $t \in (0,\infty)$, and $F = 2^{-1}(I_{-1}+I_1)$. Then*

$$\begin{aligned}
\Big|\|(F-I)^j \exp\{t(F-I)\}\| - \frac{\|\varphi_{2j}\|_1}{(2t)^j}\Big| &\le \frac{C(j)}{t^{j+1/2}}, \qquad (j \neq 0),\\
\Big||(F-I)^j \exp\{t(F-I)\}| - \frac{\|\varphi_{2j-1}\|_\infty}{(2t)^j}\Big| &\le \frac{C(j)}{t^{j+1/2}}, \qquad (j \neq 0),\\
\Big||(F-I)^j \exp\{t(F-I)\}|_0 - \frac{\|\varphi_{2j}\|_\infty}{2^j\, t^{j+1/2}}\Big| &\le \frac{C(j)}{t^{j+1}}.
\end{aligned}$$

The following proposition is an improvement of Lemma 4.9 in Čekanavičius and Roos [12], where it was assumed that $0 < p \le C < 1/2$.

Proposition 3.4 *Let $j \in \mathbb{Z}_+$, $n \in \mathbb{N}$, $p = 1-q \in (0,1)$, and $F = 2^{-1}(I_{-1}+I_1)$. Then*

$$\Big|\|(F-I)^j(I+p(F-I))^n\| - \frac{\|\varphi_{2j}\|_1}{(2np)^j}\Big| \le \frac{C(j)}{q(npq)^{j+1/2}}, \qquad (j \neq 0), \tag{92}$$

$$\Big||(F-I)^j(I+p(F-I))^n| - \frac{\|\varphi_{2j-1}\|_\infty}{(2np)^j}\Big| \le \frac{C(j)}{q(npq)^{j+1/2}}, \qquad (j \neq 0), \tag{93}$$

$$\Big||(F-I)^j(I+p(F-I))^n|_0 - \frac{\|\varphi_{2j}\|_\infty}{2^j(np)^{j+1/2}}\Big| \le \frac{C(j)}{q(npq)^{j+1}}. \tag{94}$$

Proof. We use the second bound in (38) under the assumption that $p_1 = \dots = p_n = p$ and $r = q$. Together with Lemma 3.12 and (45), we obtain, for $j \neq 0$, that

$$\Big|\|(F-I)^j(I+p(F-I))^n\| - \frac{\|\varphi_{2j}\|_1}{(2np)^j}\Big| \le T_1 + T_2,$$

where

$$\begin{aligned}
T_1 &:= \|(F-I)^{j+2}\exp\{npq(F-I)\}\|\,\|g(F)\| \le \frac{C(j)\,np^2}{(npq)^{j+3/2}} \le \frac{C(j)}{q(npq)^{j+1/2}},\\
T_2 &:= \Big|\|(F-I)^j\exp\{np(F-I)\}\| - \frac{\|\varphi_{2j}\|_1}{(2np)^j}\Big| \le \frac{C(j)}{(np)^{j+1/2}} \le \frac{C(j)}{q(npq)^{j+1/2}}.
\end{aligned}$$

Here, g is the same as in Lemma 3.2. The bound (92) is proved. Inequalities (93) and (94) are shown in the same way. Note that, for the local norm, we used (46) instead of (45). □

The next goal is Proposition 3.5 below. For its proof, we need two lemmas. The next lemma is needed in the proof of Lemma 3.14 below. Its proof is elementary but somewhat lengthy.

Lemma 3.13 *Let $j \in \mathbb{Z}_+$, $n \in \mathbb{N}$, $p = 1-q \in (0,1)$, $\widetilde{b}, y \in \mathbb{R}$, and $a = np + \widetilde{b}$. Then*

$$\begin{aligned}
&\big[(\mathrm{e}^{\mathrm{i}y}-1)^j(q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}ay}\big]''\\
&= \Big[-j(j-1)(\mathrm{e}^{\mathrm{i}y}-1)^{j-2}(q+p\mathrm{e}^{\mathrm{i}y})^n - 2j\,npq\,(\mathrm{e}^{\mathrm{i}y}-1)^j(q+p\mathrm{e}^{\mathrm{i}y})^{n-1}\\
&\quad - npq\,(\mathrm{e}^{\mathrm{i}y}-1)^j(q+p\mathrm{e}^{\mathrm{i}y})^{n-2} - (npq)^2(\mathrm{e}^{\mathrm{i}y}-1)^{j+2}(q+p\mathrm{e}^{\mathrm{i}y})^{n-2}\\
&\quad - j(j-1)(\mathrm{e}^{\mathrm{i}y}+1)(\mathrm{e}^{\mathrm{i}y}-1)^{j-1}(q+p\mathrm{e}^{\mathrm{i}y})^n - j\mathrm{e}^{\mathrm{i}y}(\mathrm{e}^{\mathrm{i}y}-1)^{j-1}(q+p\mathrm{e}^{\mathrm{i}y})^n\\
&\quad - 2j\,npq\,(\mathrm{e}^{\mathrm{i}y}-1)^{j+1}(q+p\mathrm{e}^{\mathrm{i}y})^{n-1} + 2j\widetilde{b}\mathrm{e}^{\mathrm{i}y}(\mathrm{e}^{\mathrm{i}y}-1)^{j-1}(q+p\mathrm{e}^{\mathrm{i}y})^n\\
&\quad + 2\widetilde{b}\,npq\,(\mathrm{e}^{\mathrm{i}y}-1)^{j+1}(q+p\mathrm{e}^{\mathrm{i}y})^{n-1} - npq\,(\mathrm{e}^{\mathrm{i}y}-1)^{j+1}(q+p\mathrm{e}^{\mathrm{i}y})^{n-2}\\
&\quad - \widetilde{b}^2(\mathrm{e}^{\mathrm{i}y}-1)^j(q+p\mathrm{e}^{\mathrm{i}y})^n\Big]\mathrm{e}^{-\mathrm{i}ay},
\end{aligned}$$

where the derivatives are carried out with respect to y.

Proof. We have

$$\big[(\mathrm{e}^{\mathrm{i}y}-1)^j(q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}ay}\big]''=T_1+T_2+T_3,$$

where

$$T_1=\big[(\mathrm{e}^{\mathrm{i}y}-1)^j\big]''(q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}ay},\quad T_2=2j\mathrm{i}\mathrm{e}^{\mathrm{i}y}(\mathrm{e}^{\mathrm{i}y}-1)^{j-1}\big[(q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}ay}\big]',$$
$$T_3=(\mathrm{e}^{\mathrm{i}y}-1)^j\big[(q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}ay}\big]''.$$

The term T_1 can be evaluated in the following way:

$$\begin{aligned}T_1 &= \big[-j(j-1)\mathrm{e}^{\mathrm{i}2y}(\mathrm{e}^{\mathrm{i}y}-1)^{j-2}-j\mathrm{e}^{\mathrm{i}y}(\mathrm{e}^{\mathrm{i}y}-1)^{j-1}\big](q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}ay}\\ &= \big[-j(j-1)(\mathrm{e}^{\mathrm{i}y}-1)^{j-2}-j(j-1)(\mathrm{e}^{\mathrm{i}y}+1)(\mathrm{e}^{\mathrm{i}y}-1)^{j-1}\\ &\quad -j\mathrm{e}^{\mathrm{i}y}(\mathrm{e}^{\mathrm{i}y}-1)^{j-1}\big](q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}ay}.\end{aligned}$$

For the calculation of T_2, we define a centered characteristic function $h(y)=q\mathrm{e}^{-\mathrm{i}py}+p\mathrm{e}^{\mathrm{i}qy}=(q+p\mathrm{e}^{\mathrm{i}y})\mathrm{e}^{-\mathrm{i}py}$. Then, we have

$$\begin{aligned}(q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}ay} &= \big(q\mathrm{e}^{-\mathrm{i}py}+p\mathrm{e}^{\mathrm{i}qy}\big)^n\mathrm{e}^{-\mathrm{i}\widetilde{b}y}=h^n(y)\mathrm{e}^{-\mathrm{i}\widetilde{b}y},\\ [h^n(y)]' &= nh^{n-1}(y)h'(y)=nh^{n-1}(y)(-\mathrm{i}pq\mathrm{e}^{-\mathrm{i}py}+\mathrm{i}pq\mathrm{e}^{\mathrm{i}qy})\\ &= \mathrm{i}\,npq\,h^{n-1}(y)\mathrm{e}^{-\mathrm{i}py}(\mathrm{e}^{\mathrm{i}y}-1)=\mathrm{i}\,npq\,(q+p\mathrm{e}^{\mathrm{i}y})^{n-1}\mathrm{e}^{-\mathrm{i}npy}(\mathrm{e}^{\mathrm{i}y}-1),\\ \big[h^n(y)\mathrm{e}^{-\mathrm{i}\widetilde{b}y}\big]' &= \mathrm{i}\,npq\,(q+p\mathrm{e}^{\mathrm{i}y})^{n-1}(\mathrm{e}^{\mathrm{i}y}-1)\mathrm{e}^{-\mathrm{i}ay}-\mathrm{i}\widetilde{b}(q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}ay}.\end{aligned}$$

Consequently,

$$\begin{aligned}T_2 &= -2j\,npq\,\mathrm{e}^{\mathrm{i}y}(\mathrm{e}^{\mathrm{i}y}-1)^j(q+p\mathrm{e}^{\mathrm{i}y})^{n-1}\mathrm{e}^{-\mathrm{i}ay}+2j\widetilde{b}\mathrm{e}^{\mathrm{i}y}(\mathrm{e}^{\mathrm{i}y}-1)^{j-1}(q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}ay}\\ &= \big[-2j\,npq\,(\mathrm{e}^{\mathrm{i}y}-1)^j(q+p\mathrm{e}^{\mathrm{i}y})^{n-1}-2j\,npq\,(\mathrm{e}^{\mathrm{i}y}-1)^{j+1}(q+p\mathrm{e}^{\mathrm{i}y})^{n-1}\\ &\quad +2j\widetilde{b}\mathrm{e}^{\mathrm{i}y}(\mathrm{e}^{\mathrm{i}y}-1)^{j-1}(q+p\mathrm{e}^{\mathrm{i}y})^n\big]\mathrm{e}^{-\mathrm{i}ay}.\end{aligned}$$

The term T_3 can be treated, taking into account that the second derivative of $h^n(y)$ is equal to

$$\begin{aligned}[h^n(y)]'' &= \mathrm{i}\,npq\,\big[(q+p\mathrm{e}^{\mathrm{i}y})^{n-1}\mathrm{e}^{-\mathrm{i}npy}(\mathrm{e}^{\mathrm{i}y}-1)\big]'\\ &= \mathrm{i}\,npq\,\big[(n-1)(q+p\mathrm{e}^{\mathrm{i}y})^{n-2}\mathrm{i}p\mathrm{e}^{\mathrm{i}y}\mathrm{e}^{-\mathrm{i}npy}(\mathrm{e}^{\mathrm{i}y}-1)\\ &\quad -(q+p\mathrm{e}^{\mathrm{i}y})^{n-1}\,\mathrm{i}np\,\mathrm{e}^{-\mathrm{i}npy}(\mathrm{e}^{\mathrm{i}y}-1)+(q+p\mathrm{e}^{\mathrm{i}y})^{n-1}\mathrm{e}^{-\mathrm{i}npy}\mathrm{i}\mathrm{e}^{\mathrm{i}y}\big]\\ &= -npq\,(q+p\mathrm{e}^{\mathrm{i}y})^{n-2}\mathrm{e}^{-\mathrm{i}npy}\big[(n-1)p\mathrm{e}^{\mathrm{i}y}(\mathrm{e}^{\mathrm{i}y}-1)-np(q+p\mathrm{e}^{\mathrm{i}y})(\mathrm{e}^{\mathrm{i}y}-1)\\ &\quad +(q+p\mathrm{e}^{\mathrm{i}y})\mathrm{e}^{\mathrm{i}y}\big]\\ &= -npq\,(q+p\mathrm{e}^{\mathrm{i}y})^{n-2}\mathrm{e}^{-\mathrm{i}npy}\big[np\mathrm{e}^{\mathrm{i}y}(\mathrm{e}^{\mathrm{i}y}-1)-np(\mathrm{e}^{\mathrm{i}y}-1)-np^2(\mathrm{e}^{\mathrm{i}y}-1)^2+\mathrm{e}^{\mathrm{i}y}\big]\\ &= -npq\,(q+p\mathrm{e}^{\mathrm{i}y})^{n-2}\mathrm{e}^{-\mathrm{i}npy}\big[\mathrm{e}^{\mathrm{i}y}+npq\,(\mathrm{e}^{\mathrm{i}y}-1)^2\big]\\ &= -(npq)^2(\mathrm{e}^{\mathrm{i}y}-1)^2(q+p\mathrm{e}^{\mathrm{i}y})^{n-2}\mathrm{e}^{-\mathrm{i}npy}-npq\,\mathrm{e}^{\mathrm{i}y}(q+p\mathrm{e}^{\mathrm{i}y})^{n-2}\mathrm{e}^{-\mathrm{i}npy}\\ &= -(npq)^2(\mathrm{e}^{\mathrm{i}y}-1)^2(q+p\mathrm{e}^{\mathrm{i}y})^{n-2}\mathrm{e}^{-\mathrm{i}npy}-npq\,(q+p\mathrm{e}^{\mathrm{i}y})^{n-2}\mathrm{e}^{-\mathrm{i}npy}\\ &\quad -npq\,(\mathrm{e}^{\mathrm{i}y}-1)(q+p\mathrm{e}^{\mathrm{i}y})^{n-2}\mathrm{e}^{-\mathrm{i}npy}.\end{aligned}$$

Therefore,

$$\begin{aligned}
\big[(q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}ay}\big]'' &= \big[h^n(y)\mathrm{e}^{-\mathrm{i}\widetilde{b}y}\big]'' = [h^n(y)]''\mathrm{e}^{-\mathrm{i}\widetilde{b}y} + 2[h^n(y)]'[\mathrm{e}^{-\mathrm{i}\widetilde{b}y}]' + h^n(y)[\mathrm{e}^{-\mathrm{i}\widetilde{b}y}]'' \\
&= [h^n(y)]''\mathrm{e}^{-\mathrm{i}\widetilde{b}y} + 2\mathrm{i}\,npq\,(q+p\mathrm{e}^{\mathrm{i}y})^{n-1}\mathrm{e}^{-\mathrm{i}npy}(\mathrm{e}^{\mathrm{i}y}-1)(-\mathrm{i}\widetilde{b})\mathrm{e}^{-\mathrm{i}\widetilde{b}y} \\
&\quad + h^n(y)(-\mathrm{i}\widetilde{b})^2\mathrm{e}^{-\mathrm{i}\widetilde{b}y} \\
&= \big[-(npq)^2(\mathrm{e}^{\mathrm{i}y}-1)^2(q+p\mathrm{e}^{\mathrm{i}y})^{n-2} - npq\,(q+p\mathrm{e}^{\mathrm{i}y})^{n-2} \\
&\quad - npq\,(\mathrm{e}^{\mathrm{i}y}-1)(q+p\mathrm{e}^{\mathrm{i}y})^{n-2} + 2\widetilde{b}\,npq\,(\mathrm{e}^{\mathrm{i}y}-1)(q+p\mathrm{e}^{\mathrm{i}y})^{n-1} \\
&\quad - \widetilde{b}^2(q+p\mathrm{e}^{\mathrm{i}y})^n\big]\mathrm{e}^{-\mathrm{i}ay}.
\end{aligned}$$

To get T_3, it remains to multiply the last identity by $(\mathrm{e}^{\mathrm{i}y}-1)^j$. Combining the equalities for T_1, T_2, T_3, the proof is easily completed. □

In what follows, some notation is needed. For a sequence $f:\mathbb{Z}\longrightarrow\mathbb{R}$, let $\Delta^0 f(m) = f(m)$, $(m\in\mathbb{Z})$ and

$$\Delta^j f(m) := \Delta^{j-1}f(m-1) - \Delta^{j-1}f(m), \qquad (m\in\mathbb{Z},\ j\in\mathbb{N}).$$

For $p = 1-q\in[0,1]$, $j,n\in\mathbb{Z}_+$, and $m\in\mathbb{Z}$, we set $\mathrm{bi}(m,n,p) := \mathrm{Bi}(n,p,I_1)(\{m\})$ and write $\Delta^j\mathrm{bi}(m,\,n,\,p) = (\Delta^j\mathrm{bi}(\cdot,\,n,\,p))(m)$.

Lemma 3.14 *Let $j\in\mathbb{Z}_+$, $n\in\mathbb{N}$, and $p=1-q\in(0,1)$. Let S be a set and $b:\mathbb{N}\times(0,1)\times\mathbb{R}\times S\longrightarrow\mathbb{R}$ be a bounded function. Then*

$$\sup_{z\in S}\sup_{x\in\mathbb{R}}\Big|(npq)^{(j+1)/2}\Delta^j\mathrm{bi}(m,\,n,\,p) - (-1)^j\varphi_j(x)\Big| \le \frac{C(j)}{\sqrt{npq}}, \tag{95}$$

$$\sup_{z\in S}\sup_{x\in\mathbb{R}}(1+x^2)\Big|(npq)^{(j+1)/2}\Delta^j\mathrm{bi}(m,\,n,\,p) - (-1)^j\varphi_j(x)\Big| \le \frac{C(j)}{\sqrt{npq}}, \tag{96}$$

where $m = \lfloor np + x\sqrt{npq} + b(n,p,x,z)\rfloor$.

Proof. We have $m = np + x\sqrt{npq} + \widetilde{b}$, where $\widetilde{b} = \widetilde{b}(n,p,x,z)$ is a new function with $|\widetilde{b}|\le C_1$ and C_1 is an absolute constant. Without loss of generality, we may assume that $n\ge 3$. Indeed, by using simple calculus, we see that, for $n<3$, the left-hand sides of (95) and (96) are bounded by $C(j)/\sqrt{pq}$. Further, we may assume that $m\ge 0$. In fact, if $m<0$, then $\Delta^j\mathrm{bi}(m,n,p) = 0$ and the estimates in (95) and (96) follow from the definition of $\varphi_j(x)$; note that, here, it can be used that, either $npq\le 2C_1$, or $npq\ge 2C_1$, the latter of which implies that

$$x \le \frac{-np-\widetilde{b}}{\sqrt{npq}} \le -\sqrt{npq} + \frac{C_1}{\sqrt{npq}} \le -\frac{\sqrt{npq}}{2}.$$

Let us now assume that $m\ge 0$. Then, by using the Fourier inversion formula, we obtain

$$(npq)^{(j+1)/2}\Delta^j\mathrm{bi}(m,n,p) = \frac{(npq)^{(j+1)/2}}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-\mathrm{i}my}\,(\mathrm{e}^{\mathrm{i}y}-1)^j(q+p\mathrm{e}^{\mathrm{i}y})^n\,\mathrm{d}y. \tag{97}$$

On the other hand, for $x\in\mathbb{R}$, we have

$$\begin{aligned}
(-1)^j\varphi_j(x) &= \frac{1}{2\pi}\int_{\mathbb{R}}(\mathrm{i}y)^j\,\mathrm{e}^{-\mathrm{i}xy}\,\mathrm{e}^{-y^2/2}\,\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{|y|>\pi\sqrt{npq}}(\mathrm{i}y)^j\,\mathrm{e}^{-\mathrm{i}xy}\,\mathrm{e}^{-y^2/2}\,\mathrm{d}y \\
&\quad + \frac{(npq)^{(j+1)/2}}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-\mathrm{i}my}(\mathrm{i}y)^j\exp\Big\{\mathrm{i}y(np+\widetilde{b}) - \frac{npq\,y^2}{2}\Big\}\,\mathrm{d}y.
\end{aligned} \tag{98}$$

The absolute value of the first integral in (98) is bounded by $C(j)\mathrm{e}^{-C\,npq}$. Therefore, for the proof of (95), it suffices to give a bound for the difference between the integral in (97) and the second one in (98). For this, we need some preparation. It is well-known that, for $z \in \mathbb{C}$ and $k \in \mathbb{Z}_+$,

$$\Big|\mathrm{e}^z - \sum_{m=0}^{k} \frac{z^m}{m!}\Big| \le \frac{|z|^{k+1}}{(k+1)!}\,\mathrm{e}^{\max\{\mathrm{Re}(z),\,0\}}. \tag{99}$$

Using (99), it is easily shown that, for $y \in \mathbb{R}$,

$$|(\mathrm{i}y)^j - (\mathrm{e}^{\mathrm{i}y}-1)^j| \le \sum_{m=1}^{j} |y|^{m-1}|\mathrm{e}^{\mathrm{i}y}-1-\mathrm{i}y||\mathrm{e}^{\mathrm{i}y}-1|^{j-m} \le \frac{j}{2}\,|y|^{j+1}, \tag{100}$$

$$|\mathrm{e}^{\mathrm{i}\widetilde{b}y}-1| \le C_1|y|, \tag{101}$$

and, for $|y| \le \pi$,

$$\begin{aligned}
|q\mathrm{e}^{-\mathrm{i}py} + p\mathrm{e}^{\mathrm{i}qy} - \mathrm{e}^{-pqy^2/2}| &\le q\Big|\mathrm{e}^{-\mathrm{i}py} - 1 + \mathrm{i}py - \frac{(\mathrm{i}py)^2}{2}\Big| + p\Big|\mathrm{e}^{\mathrm{i}qy} - 1 - \mathrm{i}qy - \frac{(\mathrm{i}qy)^2}{2}\Big| \\
&\quad + \Big|1 - \frac{pqy^2}{2} + \frac{(pqy^2)^2}{8} - \mathrm{e}^{-pqy^2/2}\Big| + \frac{(pqy^2)^2}{8} \\
&\le C\,pq\,|y|^3.
\end{aligned} \tag{102}$$

Further, for $|y| \le \pi$,

$$|q + p\mathrm{e}^{\mathrm{i}y}| = \Big(1 - 4pq\sin^2\Big(\frac{y}{2}\Big)\Big)^{1/2} \le \exp\Big\{-2pq\sin^2\Big(\frac{y}{2}\Big)\Big\} \le \exp\Big\{-\frac{2pqy^2}{\pi^2}\Big\}. \tag{103}$$

Using (102) and (103), for $|y| \le \pi$, we get

$$\begin{aligned}
&\Big|(q+p\mathrm{e}^{\mathrm{i}y})^n - \exp\Big\{\mathrm{i}\,npy - \frac{npq\,y^2}{2}\Big\}\Big| \\
&\le \sum_{m=1}^{n} |q+p\mathrm{e}^{\mathrm{i}y}|^{m-1}\Big|q+p\mathrm{e}^{\mathrm{i}y} - \exp\Big\{\mathrm{i}py - \frac{pqy^2}{2}\Big\}\Big|\exp\Big\{-\frac{(n-m)pq\,y^2}{2}\Big\} \\
&\le C\,n\exp\Big\{-\frac{2}{\pi^2}\,npq\,y^2\Big\}|q\mathrm{e}^{-\mathrm{i}py} + p\mathrm{e}^{\mathrm{i}qy} - \mathrm{e}^{-pqy^2/2}| \\
&\le C\,npq\,|y|^3\exp\{-C\,npq\,y^2\} \\
&\le \frac{C}{\sqrt{npq}}\exp\{-C\,npq\,y^2\}.
\end{aligned} \tag{104}$$

Finally, we need

$$\int_{-\pi}^{\pi} |y|^j \exp\{-C\,npq\,y^2\}\,\mathrm{d}y \le \frac{C(j)}{(npq)^{(j+1)/2}}. \tag{105}$$

Using the triangle inequality together with (97), (98), (100), (101), (104), and (105), the proof of (95) is easily completed. Let us now show (96). Set $a = np + \widetilde{b}$. Then, in view of (97), integrating by parts and using Lemma 3.13, we obtain

$$\begin{aligned}
&2\pi x^2(npq)^{(j+1)/2}\Delta^j \mathrm{bi}(m,n,p) \\
&\qquad = -(npq)^{(j-1)/2}\int_{-\pi}^{\pi} \big[(\mathrm{e}^{\mathrm{i}y}-1)^j(q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}ay}\big]''\mathrm{e}^{-\mathrm{i}(m-a)y}\,\mathrm{d}y \\
&\qquad = J_1 + \cdots + J_5,
\end{aligned} \tag{106}$$

where the derivatives are carried out with respect to y and

$$\begin{aligned}
J_1 &= j(j-1)\,(npq)^{(j-1)/2}\int_{-\pi}^{\pi}(\mathrm{e}^{\mathrm{i}y}-1)^{j-2}(q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}my}\,\mathrm{d}y,\\
J_2 &= 2j\,(npq)^{(j+1)/2}\int_{-\pi}^{\pi}(\mathrm{e}^{\mathrm{i}y}-1)^{j}(q+p\mathrm{e}^{\mathrm{i}y})^{n-1}\mathrm{e}^{-\mathrm{i}my}\,\mathrm{d}y,\\
J_3 &= (npq)^{(j+1)/2}\int_{-\pi}^{\pi}(\mathrm{e}^{\mathrm{i}y}-1)^{j}(q+p\mathrm{e}^{\mathrm{i}y})^{n-2}\mathrm{e}^{-\mathrm{i}my}\,\mathrm{d}y,\\
J_4 &= (npq)^{(j+3)/2}\int_{-\pi}^{\pi}(\mathrm{e}^{\mathrm{i}y}-1)^{j+2}(q+p\mathrm{e}^{\mathrm{i}y})^{n-2}\mathrm{e}^{-\mathrm{i}my}\,\mathrm{d}y,\\
J_5 &= j(j-1)\,(npq)^{(j-1)/2}\int_{-\pi}^{\pi}(\mathrm{e}^{\mathrm{i}y}+1)(\mathrm{e}^{\mathrm{i}y}-1)^{j-1}(q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}my}\,\mathrm{d}y\\
&\quad +j\,(npq)^{(j-1)/2}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}y}(\mathrm{e}^{\mathrm{i}y}-1)^{j-1}(q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}my}\,\mathrm{d}y\\
&\quad +2j\,(npq)^{(j+1)/2}\int_{-\pi}^{\pi}(\mathrm{e}^{\mathrm{i}y}-1)^{j+1}(q+p\mathrm{e}^{\mathrm{i}y})^{n-1}\mathrm{e}^{-\mathrm{i}my}\,\mathrm{d}y\\
&\quad -2j\widetilde{b}\,(npq)^{(j-1)/2}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}y}(\mathrm{e}^{\mathrm{i}y}-1)^{j-1}(q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}my}\,\mathrm{d}y\\
&\quad -2\widetilde{b}\,(npq)^{(j+1)/2}\int_{-\pi}^{\pi}(\mathrm{e}^{\mathrm{i}y}-1)^{j+1}(q+p\mathrm{e}^{\mathrm{i}y})^{n-1}\mathrm{e}^{-\mathrm{i}my}\,\mathrm{d}y\\
&\quad +(npq)^{(j+1)/2}\int_{-\pi}^{\pi}(\mathrm{e}^{\mathrm{i}y}-1)^{j+1}(q+p\mathrm{e}^{\mathrm{i}y})^{n-2}\mathrm{e}^{-\mathrm{i}my}\,\mathrm{d}y\\
&\quad +\widetilde{b}^2\,(npq)^{(j-1)/2}\int_{-\pi}^{\pi}(\mathrm{e}^{\mathrm{i}y}-1)^{j}(q+p\mathrm{e}^{\mathrm{i}y})^n\mathrm{e}^{-\mathrm{i}my}\,\mathrm{d}y.
\end{aligned}$$

Similarly,

$$2\pi x^2(-1)^j\varphi_j(x) = -\int_{\mathbb{R}}\big[(\mathrm{i}y)^j\mathrm{e}^{-y^2/2}\big]''\mathrm{e}^{-\mathrm{i}xy}\,\mathrm{d}y = \widetilde{J}_0+\cdots+\widetilde{J}_4, \tag{107}$$

where

$$\begin{aligned}
\widetilde{J}_0 &= -\int_{|y|>\pi\sqrt{npq}}\big[(\mathrm{i}y)^j\mathrm{e}^{-y^2/2}\big]''\mathrm{e}^{-\mathrm{i}xy}\,\mathrm{d}y,\\
\widetilde{J}_1 &= j(j-1)\,(npq)^{(j-1)/2}\int_{-\pi}^{\pi}(\mathrm{i}y)^{j-2}\exp\Big\{\mathrm{i}y(np+\widetilde{b})-\frac{npq\,y^2}{2}\Big\}\,\mathrm{e}^{-\mathrm{i}my}\,\mathrm{d}y,\\
\widetilde{J}_2 &= 2j\,(npq)^{(j+1)/2}\int_{-\pi}^{\pi}(\mathrm{i}y)^{j}\exp\Big\{\mathrm{i}y(np+\widetilde{b})-\frac{npq\,y^2}{2}\Big\}\,\mathrm{e}^{-\mathrm{i}my}\,\mathrm{d}y,\\
\widetilde{J}_3 &= (npq)^{(j+1)/2}\int_{-\pi}^{\pi}(\mathrm{i}y)^{j}\exp\Big\{\mathrm{i}y(np+\widetilde{b})-\frac{npq\,y^2}{2}\Big\}\,\mathrm{e}^{-\mathrm{i}my}\,\mathrm{d}y,\\
\widetilde{J}_4 &= (npq)^{(j+3)/2}\int_{-\pi}^{\pi}(\mathrm{i}y)^{j+2}\exp\Big\{\mathrm{i}y(np+\widetilde{b})-\frac{npq\,y^2}{2}\Big\}\,\mathrm{e}^{-\mathrm{i}my}\,\mathrm{d}y.
\end{aligned}$$

In view of (95), (106), and (107), we see that (96) is shown, if we give suitable bounds for $|\widetilde{J}_0|$, $|J_5|$, and $|J_\ell-\widetilde{J}_\ell|$, ($\ell\in\{1,\dots,4\}$). Clearly, $|\widetilde{J}_0|\leq C(j)\mathrm{e}^{-C\,npq}$. Further, we get $|J_5|\leq C(j)(npq)^{-1/2}$, where we used that

$$\int_{-\pi}^{\pi}|\mathrm{e}^{\mathrm{i}y}-1|^k|q+p\mathrm{e}^{\mathrm{i}y}|^n\,\mathrm{d}y\leq\frac{C(k)}{(npq)^{(k+1)/2}},\qquad(k\in\mathbb{Z}_+).$$

The latter inequality easily follows from (99), (103), and (105). With the help of the simple inequality

$$|\mathrm{e}^{z_1} - \mathrm{e}^{z_2}| \le |z_1 - z_2| \mathrm{e}^{\max\{\mathrm{Re}(z_1),\, \mathrm{Re}(z_2)\}}, \qquad (z_1, z_2 \in \mathbb{C}),$$

we obtain, for $|y| \le \pi$ and $k \in \{1,2\}$,

$$\Big|\exp\Big\{\mathrm{i}\,(n-k)py - \frac{(n-k)pq\,y^2}{2}\Big\} - \exp\Big\{\mathrm{i}\,npy - \frac{npq\,y^2}{2}\Big\}\Big| \le \frac{C}{\sqrt{npq}}\exp\{-C\,npq\,y^2\}.$$

Together with (104), for $|y| \le \pi$ and $k \in \{0,1,2\}$, this yields

$$\Big|(q + p\mathrm{e}^{\mathrm{i}y})^{n-k} - \exp\Big\{\mathrm{i}npy - \frac{npq\,y^2}{2}\Big\}\Big| \le \frac{C}{\sqrt{npq}}\exp\{-C\,npq\,y^2\}. \tag{108}$$

Now, using (100), (101), (108), and (105), it can be shown that $|J_\ell - \widetilde{J}_\ell| \le C(j)(npq)^{-1/2}$, ($\ell \in \{1, \dots, 4\}$). This completes the proof of (96). □

Proposition 3.5 *For $j \in \mathbb{Z}_+$, $n \in \mathbb{N}$, and $p = 1 - q \in (0,1)$, we have*

$$\Big|\|(I_1 - I)^j (I + p(I_1 - I))^n\| - \frac{\|\varphi_j\|_1}{(npq)^{j/2}}\Big| \le \frac{C(j)}{(npq)^{(j+1)/2}}, \quad (j \ne 0), \tag{109}$$

$$\Big||(I_1 - I)^j (I + p(I_1 - I))^n| - \frac{\|\varphi_{j-1}\|_\infty}{(npq)^{j/2}}\Big| \le \frac{C(j)}{(npq)^{(j+1)/2}}, \quad (j \ne 0), \tag{110}$$

$$\Big||(I_1 - I)^j (I + p(I_1 - I))^n|_0 - \frac{\|\varphi_j\|_\infty}{(npq)^{(j+1)/2}}\Big| \le \frac{C(j)}{(npq)^{j/2+1}}. \tag{111}$$

Proof. By using Lemma 3.14 and the fact that

$$\begin{aligned} \|(I_1 - I)^j (I + p(I_1 - I))^n\| &= \sum_{m=0}^{\infty} |\Delta^j \mathrm{bi}(m,n,p)| \\ &= \sqrt{npq} \sum_{m=0}^{\infty} \int_{(m-np)/\sqrt{npq}}^{(m+1-np)/\sqrt{npq}} |\Delta^j \mathrm{bi}(m,n,p)|\,\mathrm{d}x \\ &= \sqrt{npq} \int_{\mathbb{R}} |\Delta^j \mathrm{bi}(\lfloor np + x\sqrt{npq}\rfloor, n, p)|\,\mathrm{d}x \end{aligned}$$

and, similarly,

$$|(I_1 - I)^j (I + p(I_1 - I))^n|_0 = \sup_{x \in \mathbb{R}} |\Delta^j \mathrm{bi}(\lfloor np + x\sqrt{npq}\rfloor, n, p)|,$$

the proofs of (109) and (111) are easily completed. Inequality (110) follows from (111) and the simple equality

$$|(I_1 - I)^j (I + p(I_1 - I))^n| = |(I_1 - I)^{j-1} (I + p(I_1 - I))^n|_0,$$

for $j, n \in \mathbb{N}$ and $p = 1 - q \in (0,1)$. □

Remark 3.1 In Roos ([29], Lemma 8), using direct calculations, it was shown that

$$\Big|\|(I_1 - I)^2 (I + p(I_1 - I))^n\| - \frac{\|\varphi_2\|_1}{npq}\Big| \le \frac{C}{(npq)^2},$$

$$\Big||(I_1 - I)^2 (I + p(I_1 - I))^n|_0 - \frac{\|\varphi_2\|_\infty}{(npq)^{3/2}}\Big| \le \frac{C}{(npq)^{5/2}},$$

which is better than (109) and (111) for $j = 2$, respectively. It is not clear how to obtain analogous improvements of these bounds for $j \ne 2$.

4 Proofs of the main results

Proof of Theorem 2.1. Using (6), we obtain

$$\begin{aligned}
&\|\mathrm{GPB}(n,\,\mathbf{p},\,F)-\mathrm{EBi}(n,\,\mathbf{p},\,F;\,s)\|\\
&\qquad= \Big\|\mathrm{GPB}(n,\,\mathbf{p},\,F)\Big[I-\exp\Big\{-\sum_{j=1}^{n}\sum_{m=s+1}^{\infty}\frac{(-1)^{m+1}}{m}U_j^m\Big\}\Big]\Big\|\\
&\qquad\le \exp\Big\{\sum_{j=1}^{n}\sum_{m=s+1}^{\infty}\frac{\|U_j\|^m}{m}\Big\}-1\le\exp\Big\{\frac{1}{s+1}\sum_{j=1}^{n}\frac{\|U_j\|^{s+1}}{1-\|U_j\|}\Big\}-1,
\end{aligned}$$

where, in view of (5), under the present assumptions, $\|U_j\|\le 2|p_j-\overline{p}|/(1-2\overline{p})<1$. Now the assertion follows. □

The proofs of Theorems 2.2 and 2.3 require the following lemma. Recall that we write $D=D(n,\,\mathbf{p},\,F;\,s)$.

Lemma 4.1 *Let $F\in\mathcal{F}$ and assume that $p_{\max}\le 1/5$. Then*

$$\mathrm{GPB}(n,\,\mathbf{p},\,F)-\mathrm{EBi}(n,\,\mathbf{p},\,F;\,s)=W_1W_2W_3W_4, \tag{112}$$

where

$$\begin{aligned}
W_1 &= (F-I)^{s+1}(I+\overline{p}(F-I))^{n_{s,1}},\\
W_2 &= \sum_{j=1}^{n}\sum_{m=s+1}^{\infty}\frac{(-1)^{m+1}}{m}U_j^{m-(s+1)}(p_j-\overline{p})^{s+1}\Big(\sum_{k=0}^{\infty}(-\overline{p})^k(F-I)^k\Big)^{s+1},\\
W_3 &= \sum_{k=1}^{\infty}\frac{1}{k!}\Big(\sum_{j=1}^{n}\sum_{m=s+1}^{\infty}\frac{(-1)^{m+1}}{m}U_j^m\Big)^{k-1}(I+\overline{p}(F-I))^{n_{s,2}},\\
W_4 &= D(I+\overline{p}(F-I))^{n_{s,3}},
\end{aligned}$$

$n_{s,1}=\lfloor n/10\rfloor$, $n_{s,2}=\lfloor y_sn\rfloor$ with $y_1=0.9$, $y_2=0.36$, $y_3=0.19$, $y_4=0.11$, $y_5=y_6=y_7=\cdots=0.06$, and $n_{s,3}=n-n_{s,1}-n_{s,2}$. The following estimates are valid:

$$\|W_2\|\le\frac{\nu_{s+1}}{(s+1)(1-2\overline{p})^s(1-2(\delta+\overline{p}))},\quad \|W_3\|\le\frac{1.4}{1-2\overline{p}},\quad \|W_4\|\le 2.61.$$

Proof. Equation (112) follows from the fact that

$$\mathrm{GPB}(n,\,\mathbf{p},\,F)-\mathrm{EBi}(n,\,\mathbf{p},\,F;\,s)=\Big[\exp\Big\{\sum_{j=1}^{n}\sum_{m=s+1}^{\infty}\frac{(-1)^{m+1}}{m}U_j^m\Big\}-I\Big]D(I+\overline{p}(F-I))^n. \tag{113}$$

The norm $\|W_2\|$ is estimated as follows. Using (5) and the inequalities $\delta\le 1/5$ and $\overline{p}\le 1/5$, we obtain

$$\|W_2\|\le\frac{\nu_{s+1}}{(1-2\overline{p})^{s+1}}\sum_{m=s+1}^{\infty}\frac{1}{m}\Big(\frac{2\delta}{1-2\overline{p}}\Big)^{m-(s+1)}\le\frac{\nu_{s+1}}{(s+1)(1-2\overline{p})^s(1-2(\delta+\overline{p}))}.$$

Let us now estimate $\|W_3\|$. Using (5), (35), the fact that

$$\frac{\gamma_2}{\lambda\,\overline{q}}\le\delta\min\Big\{1,\,\frac{\delta}{4\overline{p}\,\overline{q}}\Big\}\le\frac{1}{5}\min\Big\{1,\,\frac{1}{20\overline{p}\,\overline{q}}\Big\}, \tag{114}$$

(see (12)) in combination with Stirling's formula (40) and the inequality $n/(n_{s,2}+1) \le y_s^{-1}$, we obtain

$$\begin{aligned}
(1-2\overline{p})\|W_3\| &\le \sum_{k=1}^{\infty} \frac{1}{k!} \Big\| \Big(\sum_{j=1}^{n} \sum_{m=s+1}^{\infty} \frac{(-1)^{m+1}}{m} U_j^m \Big)^{k-1} (I+\overline{p}(F-I))^{n_{s,2}+1} \Big\| \\
&\le 1 + \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \Big[\sum_{j=1}^{n} \sum_{m=s+1}^{\infty} \frac{\|U_j\|^{m-2}}{m} \|U_j^2 (I+\overline{p}(F-I))^{\lfloor (n_{s,2}+1)/k \rfloor}\| \Big]^k \\
&\le 1 + \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \Big[\sum_{m=s+1}^{\infty} \frac{1}{m} \Big(\frac{2\delta}{1-2\overline{p}} \Big)^{m-2} \frac{\gamma_2}{(1-2\overline{p})^2} \\
&\qquad\qquad \times \|(F-I)^2 (I+\overline{p}(F-I))^{\lfloor (n_{s,2}+1)/k \rfloor}\| \Big]^k \\
&\le 1 + \sum_{k=1}^{\infty} \frac{1}{(k+1)\sqrt{2\pi\, k}} \Big[\frac{n}{(n_{s,2}+1)} \frac{\mathrm{e}\sqrt{2}}{(1-2\overline{p})^2} \frac{\gamma_2}{\lambda\, \overline{q}} \sum_{m=s+1}^{\infty} \frac{1}{m} \Big(\frac{2\delta}{1-2\overline{p}} \Big)^{m-2} \Big]^k \\
&\le 1 + \sum_{k=1}^{\infty} \frac{(f_s(\overline{p}))^k}{(k+1)\sqrt{2\pi\, k}},
\end{aligned}$$

where, for $p \in [0, 1/5]$,

$$f_s(p) = \frac{\mathrm{e}\sqrt{2}}{5 y_s (1-2p)^2} \min\Big\{1,\, \frac{1}{20p\,(1-p)}\Big\} \sum_{m=s+1}^{\infty} \frac{1}{m} \Big(\frac{2}{5(1-2p)} \Big)^{m-2}.$$

Since $\overline{p} \in [0, 1/5]$, we see that $\overline{p} \le 1/2 - 1/\sqrt{5} =: c_0$ is equivalent to $20\overline{p}\,\overline{q} \le 1$. It easily follows that, for $s = 1, \dots, 5$, $f_s(\overline{p}) \le \max_{c_0 \le p \le 1/5} f_s(p) \le 0.9$. This and the fact that $f_s(\overline{p})$ is decreasing in $s \in \{5, 6, 7, \dots\}$ leads to the bound $f_s(\overline{p}) \le 0.9$ for all $s \in \mathbb{N}$, which shows that $(1-2\overline{p})\|W_3\| < 1.4$. By similar arguments, we now prove that $\|W_4\| \le 2.61$. Using (5), (35), (114), the inequality $n/n_{s,3} \le 1/(0.9 - y_s)$, and Stirling's formula (40), we obtain

$$\begin{aligned}
\|W_4\| &= \Big\| \exp\Big\{ \sum_{j=1}^{n} \sum_{m=2}^{s} \frac{(-1)^{m+1}}{m} U_j^m \Big\} (I+\overline{p}(F-I))^{n_{s,3}} \Big\| \\
&\le 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \Big\| \Big(\sum_{j=1}^{n} \sum_{m=2}^{s} \frac{(-1)^{m+1}}{m} U_j^m \Big)^k (I+\overline{p}(F-I))^{n_{s,3}} \Big\| \\
&\le 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \Big[\sum_{j=1}^{n} \sum_{m=2}^{s} \frac{\|U_j\|^{m-2}}{m} \|U_j^2 (I+\overline{p}(F-I))^{\lfloor n_{s,3}/k \rfloor}\| \Big]^k \\
&\le 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \Big[\sum_{m=2}^{s} \frac{1}{m} \Big(\frac{2\delta}{1-2\overline{p}} \Big)^{m-2} \frac{\gamma_2}{(1-2\overline{p})^2} \|(F-I)^2 (I+\overline{p}(F-I))^{\lfloor n_{s,3}/k \rfloor}\| \Big]^k \\
&\le 1 + \sum_{k=1}^{\infty} \frac{1}{\sqrt{2\pi\, k}} \Big[\frac{n}{n_{s,3}} \frac{\mathrm{e}\sqrt{2}}{(1-2\overline{p})^2} \frac{\gamma_2}{\lambda\, \overline{q}} \sum_{m=2}^{s} \frac{1}{m} \Big(\frac{2\delta}{1-2\overline{p}} \Big)^{m-2} \Big]^k \\
&\le 1 + \sum_{k=1}^{\infty} \frac{(g_s(\overline{p}))^k}{\sqrt{2\pi\, k}},
\end{aligned}$$

where, for $p \in [0, 1/5]$,

$$g_s(p) = \frac{\mathrm{e}\sqrt{2}}{5(0.9 - y_s)(1-2p)^2} \min\Big\{1,\, \frac{1}{20p\,(1-p)}\Big\} \sum_{m=2}^{s} \frac{1}{m} \Big(\frac{2}{5(1-2p)} \Big)^{m-2}.$$

Similarly to the above, we get $g_s(\overline{p}) \leq \max_{c_0 \leq p \leq 1/5} g_s(p) \leq 0.9$. This implies that, indeed, $\|W_4\| \leq 2.61$. □

Proof of Theorems 2.2 and 2.3. The proof of (14)–(17) and (19)–(20) is done with the help of Lemma 4.1, where it remains to estimate the norm terms of W_1. Here we apply Lemma 3.1, (71), (43), (44), (52), and (53). The theorems are proved. □

For the proofs of Theorems 2.4 and 2.5, we need the following lemma.

Lemma 4.2 *Let $F \in \mathcal{F}$ and assume that $p_{\max} \leq 1/5$. Then*

$$\mathrm{GPB}(n,\,\mathbf{p},\,F) - \mathrm{EBi}(n,\,\mathbf{p},\,F;\,s) + \frac{\gamma_{s+1}}{s+1}(F-I)^{s+1}(I+\overline{p}(F-I))^n = V_1+V_2+V_3+V_4, \quad (115)$$

where

$$\begin{aligned}
V_1 &= \Big[\exp\Big\{\sum_{j=1}^n \sum_{m=s+1}^\infty \frac{(-1)^{m+1}}{m} U_j^m\Big\} - I - \sum_{j=1}^n \sum_{m=s+1}^\infty \frac{(-1)^{m+1}}{m} U_j^m\Big] D(I+\overline{p}(F-I))^n, \\
V_2 &= \sum_{j=1}^n \sum_{m=s+1}^\infty \frac{(-1)^{m+1}}{m} U_j^m (D-I)(I+\overline{p}(F-I))^n, \\
V_3 &= \sum_{j=1}^n \sum_{m=s+2}^\infty \frac{(-1)^{m+1}}{m} U_j^m (I+\overline{p}(F-I))^n, \\
V_4 &= -\frac{\gamma_{s+1}}{s+1}\Big[\Big(\sum_{m=0}^\infty (-\overline{p})^m (F-I)^m\Big)^{s+1} - I\Big](F-I)^{s+1}(I+\overline{p}(F-I))^n.
\end{aligned}$$

The following estimates are valid:

$$\begin{aligned}
\|V_1\| &\leq C(s)\,\frac{\nu_{s+1}^2}{\lambda^{s+1}}, \quad \|V_2\| \leq C(s)\,\frac{\nu_{s+1}\,\gamma_2}{\lambda^{(s+3)/2}}, \\
\|V_3\| &\leq C(s)\,\frac{\nu_{s+2}}{\lambda^{(s+2)/2}}, \quad \|V_4\| \leq C(s)\,\frac{\overline{p}\,|\gamma_{s+1}|}{\lambda^{(s+2)/2}}.
\end{aligned}$$

If $F = I_1$, then

$$\begin{aligned}
|V_1|_0 &\leq C(s)\,\frac{\nu_{s+1}^2}{\lambda^{s+3/2}}, \quad |V_2|_0 \leq C(s)\,\frac{\nu_{s+1}\,\gamma_2}{\lambda^{(s+4)/2}}, \\
|V_3|_0 &\leq C(s)\,\frac{\nu_{s+2}}{\lambda^{(s+3)/2}}, \quad |V_4|_0 \leq C(s)\,\frac{\overline{p}\,|\gamma_{s+1}|}{\lambda^{(s+3)/2}}.
\end{aligned}$$

If $F \in \mathcal{S}$ is concentrated on $\mathbb{Z} \setminus \{0\}$, then

$$\begin{aligned}
\|V_1\| &\leq C(s)\,\frac{\sqrt{\sigma}\,\nu_{s+1}^2}{\lambda^{2(s+1)}}, \quad \|V_2\| \leq C(s)\,\frac{\sqrt{\sigma}\,\nu_{s+1}\,\gamma_2}{\lambda^{s+3}}, \\
\|V_3\| &\leq C(s)\,\frac{\sqrt{\sigma}\,\nu_{s+2}}{\lambda^{s+2}}, \quad \|V_4\| \leq C(s)\,\frac{\sqrt{\sigma}\,\overline{p}\,|\gamma_{s+1}|}{\lambda^{s+2}}, \\
|V_1|_0 &\leq C(s)\,\frac{\nu_{s+1}^2}{\lambda^{2s+5/2}}, \quad |V_2|_0 \leq C(s)\,\frac{\nu_{s+1}\,\gamma_2}{\lambda^{s+7/2}}, \\
|V_3|_0 &\leq C(s)\,\frac{\nu_{s+2}}{\lambda^{s+5/2}}, \quad |V_4|_0 \leq C(s)\,\frac{\overline{p}\,|\gamma_{s+1}|}{\lambda^{s+5/2}}.
\end{aligned}$$

Proof. Equality (115) is easily shown with the help of (113). The proofs of the norm estimates are similar to those of Lemma 4.1 and therefore omitted. For the local norm estimates, we additionally used (36) and (53). □

Proof of Theorems 2.4 and 2.5. By the triangle inequality, Lemma 4.2, and (109), we obtain

$$\begin{aligned}
&\Big|\|\mathrm{GPB}(n,\,\mathbf{p},\,I_1)-\mathrm{EBi}(n,\,\mathbf{p},\,I_1;\,s)\|-\frac{c_{s+1}^{(1)}|\gamma_{s+1}|}{(\lambda\,\overline{q})^{(s+1)/2}}\Big|\\
&\le\ \Big\|\mathrm{GPB}(n,\,\mathbf{p},\,I_1)-\mathrm{EBi}(n,\,\mathbf{p},\,I_1;\,s)+\frac{\gamma_{s+1}}{s+1}(I_1-I)^{s+1}(I+\overline{p}(I_1-I))^n\Big\|\\
&\quad+\frac{|\gamma_{s+1}|}{s+1}\Big|\|(I_1-I)^{s+1}(I+\overline{p}(I_1-I))^n\|-\frac{\|\varphi_{s+1}\|_1}{(\lambda\,\overline{q})^{(s+1)/2}}\Big|\\
&\le\ C(s)\Big(\frac{\nu_{s+1}^2}{\lambda^{s+1}}+\frac{\nu_{s+1}\,\gamma_2}{\lambda^{(s+3)/2}}+\frac{\nu_{s+2}+|\gamma_{s+1}|}{\lambda^{(s+2)/2}}\Big)\\
&\le\ C(s)\frac{\nu_{s+1}}{\lambda^{(s+1)/2}}\Big(\frac{\nu_{s+1}}{\lambda^{(s+1)/2}}+\frac{\gamma_2}{\lambda}+\frac{1}{\sqrt{\lambda}}\Big).
\end{aligned}$$

From (12), it follows that $\gamma_2/\lambda \le 1$, which leads to

$$\frac{\nu_{s+1}}{\lambda^{(s+1)/2}} \le \Big(\frac{\gamma_2}{\lambda}\Big)^{(s+1)/2} \le \frac{\gamma_2}{\lambda}.$$

Together with the estimate above, (21) follows. The remaining inequalities are similarly shown by using Lemma 4.2 and Propositions 3.5 and 3.4. □

Proof of Theorem 2.6. Let $F \in \mathcal{S}$. We use the simple fact that $\mathrm{Bi}(N,\,\widetilde{p},\,F) = \mathrm{GPB}(n,\,\widetilde{\mathbf{p}},\,F)$, where $\widetilde{\mathbf{p}} = (\widetilde{p}_1,\dots,\widetilde{p}_n)$. Hence

$$\mathrm{GPB}(n,\,\mathbf{p},\,F)-\mathrm{Bi}(N,\,\widetilde{p},\,F) = W_1+W_2+W_3$$

with

$$W_1 = \mathrm{GPB}(n,\,\mathbf{p},\,F)-\mathrm{EBi}(n,\,\mathbf{p},\,F;\,2),\quad W_2=\mathrm{EBi}(n,\,\mathbf{p},\,F;\,2)-\mathrm{EBi}(n,\,\widetilde{\mathbf{p}},\,F;\,2),$$
$$W_3 = \mathrm{EBi}(n,\,\widetilde{\mathbf{p}},\,F;\,2)-\mathrm{GPB}(n,\,\widetilde{\mathbf{p}},\,F).$$

Using (16), we get $|W_1| \le \nu_3/\lambda^3$ and $|W_3| \le \widetilde{\nu}_3/\lambda^3$. The norm estimate $|W_2| \le |\nu_2-\widetilde{\nu}_2|/\lambda^2$ can be estimated by similar arguments as in the proof of Lemma 4.1. □

Proof of Theorem 2.7. The assertions easily follow from the first bound in (38) with $r = 1-\sqrt{\theta}$, (34), (91), (45), and (46). Indeed, for $F \in \mathcal{F}$, it can be used that

$$\mathrm{GPB}(n,\,\mathbf{p},\,F)-\exp\{\lambda(F-I)\} = (F-I)^2\exp\{r\lambda(F-I)\}g(F),$$

where g is defined as in Lemma 3.2. □

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References

[1] T.V. Arak, A.Yu. Zaĭtsev, Uniform limit theorems for sums of independent random variables, *Proc. Steklov Inst. Math.*, **174**(1) (1988) (transl. from Russian).

[2] A.D. Barbour, O. Chryssaphinou, Compound Poisson approximation: a user's guide, *Ann. Appl. Probab.*, **11**, 964–1002 (2001).

[3] A.D. Barbour, L. Holst, S. Janson, *Poisson Approximation*, Clarendon Press, Oxford (1992).

[4] A.D. Barbour, A. Xia, Poisson perturbations. *ESAIM: Probab. Statist.*, **3**, 131–150 (1999).

[5] A. Bikelis, Asymptotic expansions for distributions of statistics, in *Proceedings of the 36th Conf. of the Lith. Math. Soc., Vilnius, June 22–23, 1995*, R. Kudžma, V. Mackevičius (Eds.), Vilnius University Press, (1996), pp. 5–28 (in Russian).

[6] K.A. Borovkov, D. Pfeifer, Pseudo-Poisson approximation for Markov chains, *Stochastic Proc. Appl.*, **61**, 163–180 (1996).

[7] V. Čekanavičius, On smoothing properties of compound Poisson distributions, *Lith. Math. J.*, **35**, 121–135 (1995).

[8] V. Čekanavičius, Asymptotic expansions in the exponent: a compound Poisson approach, *Adv. Appl. Prob.*, **29**, 374–387 (1997).

[9] V. Čekanavičius, Asymptotic expansions for compound Poisson measures. *Lith. Math. J.*, **37**, 320–336 (1997).

[10] V. Čekanavičius, On multivariate compound distributions, *Theory Probab. Appl.*, **47**, 493–506 (2002).

[11] V. Čekanavičius, B. Roos, Two-parametric compound binomial approximations, *Lith. Math. J.*, **44**, 354–373 (2004).

[12] V. Čekanavičius, B. Roos, *Compound Binomial Approximations*, Preprint, revised version, (accepted by *Ann. Inst. Statist. Math.*) (2005).

[13] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York (1978).

[14] J. Dhaene, N. De Pril, On a class of approximative computation methods in the individual risk model, *Insur. Math. Econ.*, **14**, 181–196 (1994).

[15] W. Ehm, Binomial approximation to the Poisson binomial distribution, *Statist. Probab. Lett.*, **11**, 7–16 (1991).

[16] C.-G. Esseen, Fourier analysis of distribution functions. A mathematical study of the Laplace–Gaussian law, *Acta Math.*, **77**, 1–125 (1945).

[17] W. Feller, *An Introduction to Probability Theory and Its Applications I*, Wiley, New York (1968).

[18] W. Hengartner, R. Theodorescu, *Concentration Functions*, Academic Press, New York (1973).

[19] C. Hipp, Improved approximations for the aggregate claims distribution in the individual model, *ASTIN Bull.*, **16**, 89–100 (1986).

[20] I.A. Ibragimov, É.L. Presman, On the rate of approach of the distributions of sums of independent random variables to accompanying distributions, *Theory Probab. Appl.*, **18**, 713–727 (1973).

[21] A. Kolmogorov, Sur les propriétés des fonctions de concentrations de M.P. Lévy, *Ann. Inst. Henri Poincaré*, **16**, 27–34 (1958).

[22] P. Kornya, Distribution of aggregate claims in the individual risk theory model. *Society of Actuaries: Transactions*, **35**, 823–858 (1983).

[23] J. Kruopis, Precision of approximation of the generalized binomial distribution by convolutions of Poisson measures, *Lith. Math. J.*, **26**, 37–49 (1986).

[24] L. Le Cam, On the distribution of sums of independent random variables, in *Bernoulli, Bayes, Laplace, Anniversary Volume*, J. Neyman, L. Le Cam (Eds.), Springer, New York, (1965), pp. 179–202.

[25] L. Le Cam, *Asymptotic Methods in Statistical Decision Theory*, Springer, New York (1986).

[26] É.L. Presman, Approximation of binomial distributions by infinitely divisible ones, *Theory Probab. Appl.*, **28**, 393–403 (1983).

[27] É.L. Presman, Approximation in variation of the distribution of a sum of independent Bernoulli variables with a Poisson law, *Theory Probab. Appl.*, **30**, 417–422, (1985).

[28] B.A. Rogozin, An estimate for concentration functions, *Theory Probab. Appl.*, **6**, 94–97 (1961).

[29] B. Roos, Binomial approximation to the Poisson binomial distribution: The Krawtchouk expansion, *Theory Probab. Appl.*, **45**, 258–272 (2000).

[30] B. Roos, Sharp constants in the Poisson approximation, *Statist. Probab. Lett.*, **52**, 155–168 (2001).

[31] B. Roos, Multinomial and Krawtchouk approximations to the generalized multinomial distribution, *Theory Probab. Appl.*, **46**, 103–117 (2001).

[32] B. Roos, On Hipp's compound Poisson approximations via concentration functions, *Bernoulli*, **11**, 533–557 (2005).

[33] S.Ya. Shorgin, Approximation of a generalized binomial distribution, *Theory Probab. Appl.*, **22**, 846–850 (1977).

[34] S.Y.T. Soon, Binomial approximation for dependent indicators, *Statist. Sinica*, **6**, 703–714 (1996).