Niven's irrationality method revisited

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In 1947, Ivan Niven published his famous note "A simple proof that π is irrational" [4]. The key of this proof is the use of sums of different derivatives of Legendre polynomials in order to construct a sequence (P_n) of polynomials with integer coefficients and of degree $\leq n$ fulfilling $0 < |P_n(\pi)| < 1/n!$. If, in this setting, we suppose $\pi = a/b$ to be a rational number, then $b^n P_n(\pi)$ would be an integer number with $0 < |b^n P_n(\pi)| < 1$ for all large n. This is of course impossible and hence π is irrational.

Soon after, Iwamoto [3] and Butlewski [2] exploited variations of Niven's method and constructed other approximation polynomials in order to get simple irrationality proofs for π^2, resp. e^k for any integer $k \neq 0$. In all cases the used polynomials seem to appear from nowhere and to show that they are actually in $\mathbb{Z}[x]$ is more or less tricky.

In this note we take a new look at the classic analytic irrationality proofs for π^2 and the integer powers of e, showing that the required approximation polynomials are generated by one single integral expression. Our approach makes it obvious how the polynomials come into existence, why they have integer coefficients and that the irrationality proofs for π, π^2 and e^k are only different special cases derived from the same general formula.

We start by studying the integral

$$I_k(z) := z^{k+1} \int_0^1 t^k e^{zt} dt$$

for $z \in \mathbb{C}$ and $k \in \mathbb{N}_0$. With partial integration we get the recursive formula

$$I_k(z) = z^k e^z - k I_{k-1}(z) \tag{1}$$

and considering $I_0(z) = e^z - 1$ we see (by induction over k) that for every $k \in \mathbb{N}_0$ there is a poynomial $r_k(z) \in \mathbb{Z}[z]$ of degree k such that

$$I_k(z) = r_k(z) e^z - (-1)^k k!\,.$$

Let $p_n(t) = t^n(1-t)^n/n!$ denote the Legendre polynomial of degree $2n$. Then it is well-known (and easily seen) that $p_n^{(n)}(t)$ is a polynomial of degree n and with exclusively integer coefficients. That is why the integral

$$J_n(z) := z^{n+1} \int_0^1 p_n^{(n)}(t) e^{zt} dt \tag{2}$$

is the sum of integer multiples of the integrals $z^n I_0(z), z^{n-1} I_1(z), \ldots, I_n(z)$. Hence there are two polynomials $Q_n(z), R_n(z) \in \mathbb{Z}[z]$ of degree $\leq n$ with

$$J_n(z) = Q_n(z) + R_n(z) e^z. \tag{3}$$

On the other hand, by repeatedly using partial integration we can transform the integral $J_n(z)$ into

$$J_n(z) = (-1)^n \frac{z^{2n+1}}{n!} \int_0^1 t^n (1-t)^n e^{zt} dt. \tag{4}$$

Since $0 \leq t(1-t) \leq 1/4$ for all $t \in [0,1]$ we get

$$|J_n(z)| \leq |z| e^{\mathrm{Re}(z)} \frac{|z/2|^{2n}}{n!}. \tag{5}$$

Now suppose that z and e^z are rational numbers or rational multiples of i, that is, z is of the form a/b or ia/b and e^z is of the form c/d or ic/d with $a, c \in \mathbb{Z}$ and $b, d \in \mathbb{N}$. Then $db^n J_n(z) \to 0$ as $n \to \infty$ and

$$db^n J_n(z) = db^n \big(Q(z) + R(z) e^z \big) \in \mathbb{Z} + i\mathbb{Z},$$

i.e. a Gaussian integer. From this we obtain the following

Proposition. *If $J_n(z)$ does not eventually vanish then not both of z and e^z can be rational numbers or rational multiples of i.*

If $z \neq 0$ is a real number, then obviously $J_n(z)$ is nonzero, since the integral in (4) is positive. Thus, if $x \neq 0$ is rational, the Proposition implies the irrationality of e^x and if $x \neq 1$ is a positive rational number, the Proposition implies the irrationality of $\ln(x)$.

For $z = i\pi$ we have

$$\mathrm{Im}\Big(\int_0^1 t^n (1-t)^n e^{zt} dt \Big) = \int_0^1 t^n (1-t)^n \sin(\pi t) dt > 0 \tag{6}$$

and in particular $J_n(i\pi) \neq 0$. From $e^{i\pi} = -1$ and the Proposition we obtain the irrationality of π. Moreover, since $\cos(s) = -\cos(\pi - s)$, the real part of the integral in (6) vanishes and we obtain from (4) that $J_n(i\pi)$ is real. If we denote the coefficients of $(Q_n - R_n)(z) \in \mathbb{Z}[z]$ by $c_{n,0}, \ldots, c_{n,n}$ then

$$0 \neq J_n(i\pi) = (Q_n - R_n)(i\pi) = \mathrm{Re}(Q_n - R_n)(i\pi) = \sum_{\nu=0}^{\lfloor n/2 \rfloor} (-1)^\nu c_{n,2\nu} \pi^{2\nu}.$$

Using (5) it is easily seen that also π^2 is irrational.

Remark: (3) and (4) imply that $-Q_n/R_n$ is the (n,n)-Padé approximant of e^z (cf. [1, p. 318]). By explicitly computing the approximation polynomials used by Iwamoto, one then sees that these actually equal $Q_n - R_n$. The same can be observed regarding Niven's polynomials and $\mathrm{Re}(Q_n + iR_n)$ which correspond to the case $z = i\pi/2$.

References

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