Faster Separation of 1-Wheel Inequalities by Graph Products

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Abstract
Using graph products we present an $O(|V|^2|E| + |V|^3 \log |V|)$ separation algorithm for the nonsimple 1-wheel inequalities by Cheng and Cunningham (1997) of the stable set polytope, which is faster than their $O(|V|^4)$ algorithm.

There are two ingredients for our algorithm. The main improvement stems from a reduction of separation problem to multiple shortest path problems in an auxiliary graph having only $6|V|$ vertices and $9|E|$ arcs, thereby preserving low sparsity. Then Johnson’s algorithm can be applied exploiting that preserved sparsity of the original graph which is maintained in the auxiliary graph.

In contrast, Cheng and Cunningham’s auxiliary graph is by construction dense, $|E| = O(|V|^2)$, so application of Johnson’s algorithm provides no large improvement.

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1 INTRODUCTION

Many important application problems contain large subproblems of the following binary packing type:

\[
\begin{align*}
\max & \quad c^\top x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \in \{0, 1\}^n
\end{align*}
\]

where \((A, b)\) is a matrix of nonnegative integers. In the process of solving \((\text{BPP})\) by a branch-and-cut algorithm, it is for the cutting-part helpful to associate with it Padberg’s conflict/intersection graph, see \([1, 2]\). Let branch-and-cut algorithm, it is for the cutting-part helpful to associate with it Padberg’s

1. Introduction

Let \(G = (V, E)\) be a simple connected graph with \(|V| = n \geq 2\) and \(|E| = m\). A subset of \(V\) is called stable if it does not contain adjacent vertices of \(G\). The incidence vector of a set \(N \subseteq V\) is \(\chi^N \in \{0, 1\}^V\) such that \(\chi^N_v = 1\) if \(v \in N\) and otherwise \(\chi^N_v = 0\). The stable set polytope of \(G\), denoted by \(\text{STAB}(G)\), is the convex hull of incidence vectors of stable sets of \(G\). Some well-known valid inequalities for \(\text{STAB}(G)\) include the trivial inequalities \((x_v \geq 0\ for\ v \in V)\), the odd cycle inequalities \((\sum_{v \in C} x_v \leq k\ where\ C\ is\ the\ vertex-set\ of\ an\ odd\ cycle\ of\ length\ 2k + 1)\), and the clique inequalities \((\sum_{v \in K} x_v \leq 1\ where\ K\ induces\ a\ clique)\). A clique inequality is called edge inequality if the clique has just two vertices. Let \(\text{ESTAB}(G) := \{x \in [0, 1]^V; x_u + x_v \leq 1\ \forall uv \in E\}\) and \(\text{CSTAB}(G) := \{x \in \text{ESTAB}(G); x\ fulfills\ the\ odd\ cycle\ inequalities\}\).

The separation problem for a class \(C\) of valid inequalities for a class of polytopes \(P\) is: Given \(x^* \in P\), does \(x^*\) violate one of the inequalities in \(C\)? If the answer is yes, exhibit such an inequality. Solving this problem is important to use the inequalities in a branch-and-cut approach for maximizing a linear function over some general integer program or \(\text{STAB}(G)\). (See, for example, Barahona et al. \[3\] and Nemhauser and Sigismondi \[4\].) Furthermore, if the separation problem for \(C\) is solvable in polynomial time, then the linear optimization problem over \(C\) can be solved in polynomial time, see Grötschel et al. \[5\]. The separation problem for \(C = \{\text{trivial and edge inequalities}\}\) can obviously be solved in \(O(m)\) time. If \(x^*\) satisfies the trivial and edge inequalities, then one can decide whether \(x^*\) violates an odd cycle inequality in polynomial time, as was first observed by Grötschel and Pulleyblank \[6\], Grötschel et al. \[5\]. Odd cycle inequalities can be separated by \(n\) applications of the fast Dijkstra algorithm by Fredman and Tarjan \[7\] in time \(O(nm + n^2 \log n)\). Hence the separation problem for the trivial, the edge, and the odd cycle inequalities can be solved in the same time.

Cheng and Cunningham \[8, 9\] describe a way to separate the 1-wheel inequalities in time \(O(n^4)\). They achieve this, by reducing the separation problem to multiple shortest path problems in dense graphs on \(O(n)\) vertices.

In the present study we reduce the complexity of the separation problem of 1-wheels down to \(O(n^2 m + n^3 \log n)\). As stable set problems often originate from the conflict graphs...
of more general integer programs, and as these conflict graphs tend to be sparse, this faster algorithm is important for practical applications.

In contrast to Cheng and Cunningham’s approach we construct a new auxiliary graph that is the categorical product of the original graph and a gadget on just 6 vertices; separation boils down to shortest path problems in this auxiliary graph solved by Johnson’s algorithm. As the runtime of Johnson’s algorithm depends on the number of edges of the auxiliary graph, which for our construction is just a constant multiple of the original number of edges, our approach is able to exploit sparsity of the original graph.

Plain application of Johnson’s algorithm to Cheng and Cunningham’s auxiliary graph can not achieve a comparable speed-up, as their auxiliary graph is dense by construction.

Our approach will be to consider a least-weight wheel problem and its reduction to a shortest path problem in a that product graph. Then we show that the separation problem reduces to the least-weight wheel problem.

2. Wheels

Cheng and Cunningham [8, 9] consider a wheel with \((2k + 1)\) vertices and hub \(h\) (for
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an example of a wheel on 5 vertices and hub, see Figure 1(a) and its subdivisions, see Figure 1(b). Let \( l_1, \ldots, 2k' + 1 \) be the rim where the spoke ends are \( l_1 \) up to \( l_{2k+1} \), ordered so that \( 1 = l_1 < l_2 < \cdots < l_{2k+1} \leq 2k' + 1 \). Denote the spoke paths connecting \( h \) to some \( l_i \) by \( P_i \) and their subpaths that exclude both ends by \( P_i' \). With \( |P_i'| \) we denote the number of vertices in \( P_i \). Let the interior of the spoke paths be pairwise disjoint and let the interior of the spoke paths be disjoint to the rim. A wheel has to fulfill additionally the condition that the cycles \( h, P_i, l_i, l_{i+1}, \ldots, l_{i+1}, P_i', h \) are odd for \( i = 1, 2, \ldots, 2k+1 \); for a complete specification we denote it by \( W(h; k'; l_1, l_2, \ldots, l_{2k+1}; P_1, P_2, \ldots, P_{2k+1}) \).

Let \( \mathcal{E} \) be the set of the \( l_i \) for which the paths \( P_i \) have an even number of edges, and let \( \mathcal{O} \) be the set of remaining spoke ends. Cheng and Cunningham [8, 9] show that the inequalities

\[
kx_h + \sum_{i=1}^{2k' + 1} x_i + \sum_{i \in \mathcal{E}} x_i + \sum_{i=1}^{2k' + 1} x(P_i') \leq k' + \frac{|\mathcal{E}| + \sum_{i=1}^{2k' + 1} |P_i'|}{2} \quad (I_A)
\]

\[
(k + 1)x_h + \sum_{i=1}^{2k' + 1} x_i + \sum_{i \in \mathcal{O}} x_i + \sum_{i=1}^{2k' + 1} x(P_i') \leq k' + \frac{|\mathcal{O}| + 1 + \sum_{i=1}^{2k' + 1} |P_i'|}{2} \quad (I_B)
\]

are valid and they give sufficient conditions for them to induce facets. (Here we use \( x(P) \) for a walk \( P = v_0 - \cdots - v_k + 1 \) as a shorthand for \( \sum_{i=1}^{k+1} x_{v_i} \).)

**Proposition 1** ([8, Prop. 2.2]). Let \( \sum_{i=1}^{n} a_i x_i \leq a_0 \) be a valid inequality for \( \text{STAB}(G) \) and let \( v_1 \) and \( v_2 \) be two nonadjacent vertices of \( G \). If \( H \) is obtained from \( G \) by identifying \( v_1 \) and \( v_2 \) to a single vertex \( v_{1,2} \), then \( (a_{v_1} + a_{v_2})x_{v_{1,2}} + \sum_{i=1}^{n} a_i x_i \leq a_0 \) is valid for \( \text{STAB}(H) \).

Therefore, when speaking of general or nonsimple wheels we will permit the identification of nonadjacent vertices where the coefficient of the new vertex is the sum of the coefficients of the identified vertices. The next example motivates how to avoid \( \mathcal{E} \) for \( I_A \) and \( \mathcal{O} \) for \( I_B \) by going nonsimple.

**Example 2.** Consider the simple wheel of Figure 2(a) with \( \mathcal{E} = \{1 \} \) and its \( I_A \) inequality \( 1x_0 + \sum_{i=1}^{6} x_i + x_1 = 1x_0 + \sum_{i=2}^{6} x_i + 2x_1 \leq 3 \). The simple wheel in Figure 2(b) has the \( I_A \) inequality \( x_0 + \sum_{i=2}^{8} x_i \leq 3 \) with \( \mathcal{E} = \emptyset \). Now, by identifying the two vertices 7 and 8, we obtain the nonsimple wheel of Figure 2(c) and the inequality according to Lemma 1 is \( x_0 + \sum_{i=2}^{6} x_i + 2x_7,8 \leq 3 \) which is the same (when relabeling the vertex \( \{7,8\} \)) as the original inequality that involved \( \mathcal{E} \not= \emptyset \).

More formally we obtain:

**Lemma 3.** For a wheel and its nonsimple \( I_A \) inequality we can assume without loss of generality \( \mathcal{E} = \emptyset \).

**Proof.** Given a wheel with \( v_\ell \in \mathcal{E} \) and its \( I_A \) inequality \( I \). Now consider the wheel resulting from contracting the edge of \( P_\ell \) incident with \( v_\ell \) and then subdividing the two rim edges incident with \( v_\ell \) once (the resulting vertices are called \( v_1 \) and \( v_2 \)). Notice that another wheel results, \( |\mathcal{E}| \) decreases by one, \( |P_\ell| \) decreases by one and \( k' \) increases by one; so all together, we obtain a new wheel with \( \mathcal{E}' = \mathcal{E} \setminus \{v_\ell\} \). The right hand side of the new
\[
\begin{align*}
(a) & \text{ A simple wheel with } E = \{1\} \\
(b) & \text{ A simple wheel with } E = \emptyset. \\
(c) & \text{ A nonsimple wheel with } E = \emptyset \text{ and spoke ends } 3, 4, 6.
\end{align*}
\]

Figure 2: Getting away with \( E = \emptyset \) for \( I_A \) inequality \( I' \) equals the right hand side of \( I \). Now if the two new vertices \( v_i, v_j \) are identified (hence the coefficient of \( v_{i,j} \) becomes two) then the inequality \( I \) results, now represented as a nonsimple wheel \( I_A \) inequality \( I'' \) with \( |E''| < |E| \).

Since Cheng and Cunningham [9, Thm. 4.2] proved that for simple wheels the inequality \( I_B \) can induce facets only if the odd spokes have length at least 3, it is natural to restrict ourselves to these.

**Lemma 4.** For a wheel without odd spokes of length < 3 its \( I_B \) inequality is representable by another \( I_A \) inequality with \( O = \emptyset \).

As the proof is analogous to that of Lemma 3 we omit the repetition. In the sequel we will assume \( E = \emptyset \) for \( I_A \) and \( O = \emptyset \) for \( I_B \) permitting more concise notation. For \( I_A \) we consider those simple wheels \( W(h; k; l_1, l_2, \ldots, l_{2k+1}; P_{l_1}, P_{l_2}, \ldots, P_{l_{2k+1}}) \) with \( E = \emptyset \) and call them plain odd \((2k' + 1, 2k + 1)\)-wheels or shorter plain odd wheels. Clearly they fulfill \( l_{j+1} - l_j \) is odd for \( j = 1, \ldots, 2k \) and that all spoke paths are odd. Similarly, for \( I_B \) we consider those simple wheels with \( O = \emptyset \) and call them plain even \((2k' + 1, 2k + 1)\)-wheels. Clearly they fulfill \( l_{j+1} - l_j \) is odd for \( j = 1, \ldots, 2k \) and all spoke paths are even. So we obtain for an odd wheel \( W(h; k'; l; 1 = l_1 < l_2 < \cdots < l_{2k+1} < 2k' + 1; P_{l_1}, P_{l_2}, \ldots, P_{l_{2k+1}}) \) the following form of inequality \( (I'_A) \):

\[
kx_h + \sum_{i=1}^{2k'+1} x_i + \sum_{i=1}^{2k+1} x(P_{l_i}) \leq k' + \frac{\sum_{i=1}^{2k+1} |P_{l_i}|}{2}, \quad (I'_A)
\]
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and for an even wheel the inequality $(I'_B)$ simplifies to

$$(k + 1)x_h + \sum_{i=1}^{2k'+1} x_i + \sum_{i=1}^{2k+1} x(\hat{P}_i) \leq k' + \frac{1 + \sum_{i=1}^{2k+1} |\hat{P}_i|}{2},$$  \hspace{1cm} (I'_B)$$

Because of Proposition 1, we will focus on general (that is, possibly nonsimple) plain odd/even $(2k' + 1, 2k + 1)$-wheels. Define

$W_A \text{STAB}(G) := \{x \in \text{CSTAB}(G): x \text{ fulfills all plain nonsimple } I'_A \text{-inequalities} \}$

$W_B \text{STAB}(G) := \{x \in \text{CSTAB}(G): x \text{ fulfills all plain nonsimple } I'_B \text{-inequalities} \}$

3. Separation of $I'_A$ and $I'_B$

Consider the following form of the four-fold of $(I'_A)$ for an odd $(2k' + 1, 2k + 1)$-wheel $W(h; k'; k; 1 = l_1 < l_2 < \cdots < l_{2k+1} \leq 2k' + 1; P_{l_1}, P_{l_2}, \ldots, P_{l_{2k+1}})$:

$$2 - 2x_h + 2(4k + 2)x_h + 4 \sum_{i=1}^{2k'+1} x_i + 4 \sum_{i=1}^{2k+1} x(\hat{P}_i) \leq (4k' + 2) + 2 \sum_{i=1}^{2k+1} |\hat{P}_i|$$

Here and henceforth we identify the indices of spoke ends modulo $2k + 1$ so that index $2k + 2$ is identified with 1, index $2k + 3$ is identified with 2 etc; reshuffling yields:

$$2 - 2x_h \leq \sum_{j=1}^{2k+1} \left( -2x_h + (|\hat{P}_{l_j}| - 2x(\hat{P}_{l_j})) + (|\hat{P}_{l_{j+1}}| - 2x(\hat{P}_{l_{j+1}})) \right.$$

$$\left. + 2 \sum_{i=l_j}^{l_{j+1}-1} (1 - x_i - x_{i+1}) \right). \hspace{1cm} (1)$$

Similarly, one obtains for an even wheel and $(I'_B)$:

$$2x_h \leq \sum_{j=1}^{2k+1} \left( -2x_h + (|\hat{P}_{l_j}| - 2x(\hat{P}_{l_j})) + (|\hat{P}_{l_{j+1}}| - 2x(\hat{P}_{l_{j+1}})) \right.$$

$$\left. + 2 \sum_{i=l_j}^{l_{j+1}-1} (1 - x_i - x_{i+1}) \right). \hspace{1cm} (2)$$

Call the right hand sides of equations (1) and (2) the weights of the odd/even wheel with respect to $x$. Notice that except for the fact that in (1) the $P$-paths are odd and in (2) they are even, the functional form of the weight is the same. Now we prove for given $\bar{x}, h, \text{rim and spoke ends a property of the spoke walks } P_{l_j} \text{ in a most violated odd/even wheel inequality.}$

Lemma 5 (Theorem 3.5 [9]). For a given graph $G$ and $\bar{x} \in \text{ESTAB}(G)$ fix a hub $h$, an odd cycle $v_1, v_2, \ldots, v_{2k'+1}$, and spoke ends $1 = l_1 < l_2 < \cdots < l_{2k+1} \leq 2k' + 1$. Among
all odd wheels with this hub and rim and these spoke ends consider one (with some $P_{l_j}$) such that

$$2 - 2\bar{x}_h \leq \sum_{j=1}^{2k+1} \left( -2\bar{x}_h + (|\hat{P}_{l_j}|-2\bar{x}(\hat{P}_{l_j})) + (|\hat{P}_{l_j+1}|-2\bar{x}(\hat{P}_{l_j+1})) \right)$$

$$+ 2 \sum_{i=l_j}^{l_j+1-1} (1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}}) \quad [l_A]$$

is most violated, that is the right hand side is minimal. Then $P_{l_j}$ is a shortest odd walk from $h$ to $v_j$ with respect to the edge weights $1 - \bar{x}_u - \bar{x}_v$, which are nonnegative because of $\bar{x} \in \text{CSTAB}(G)$.

Similarly for a most violated inequality $[l_A]$ for fixed hub and spoke ends it follows that $P_{l_j}$ is a shortest even walk (among all even walks with at least 2 edges) from $h$ to $v_j$ with respect to the edge weights $1 - \bar{x}_u - \bar{x}_v$.

Proof. Consider the odd wheel $W(h;k';k;1 = l_1 < l_2 < \cdots < l_{2k+1} \leq 2k' + 1; P_{l_1}, P_{l_2}, \ldots, P_{l_{2k+1}})$ and suppose $P'_{l_j}$ is shorter than $P_{l_j}$, that is $\sum_{uv \in E(P'_{l_j})} (1 - \bar{x}_u - \bar{x}_v) < \sum_{uv \in E(P_{l_j})} (1 - \bar{x}_u - \bar{x}_v)$ or equivalently

$$2 + |\hat{P}'_{l_j}| - 2\bar{x}(\hat{P}'_{l_j}) - \bar{x}_h - \bar{x}_{v_j} < 2 + |\hat{P}_{l_j}| - 2\bar{x}(\hat{P}_{l_j}) - \bar{x}_h - \bar{x}_{v_j}.$$ 

This implies $|\hat{P}'_{l_j}| - 2\bar{x}(\hat{P}'_{l_j}) < |\hat{P}_{l_j}| - 2\bar{x}(\hat{P}_{l_j})$. But now the odd wheel $W'$ that results from $W$ by replacing $P_{l_j}$ by $P'_{l_j}$ has less weight than the supposedly minimum weight wheel $W$ and therefore is more violated than $W$. Contradiction! The argument for $[l_B]$ is analogous.

From now on we assume for some fixed $\bar{x} \in \text{ESTAB}(G)$ that we have computed shortest odd walks $P_{l_k}^{1,h,k}$ and shortest even walks $P_{l_k}^{0,h,k}$ (the latter having at least two edges) with respect to edge weights $(1 - \bar{x}_u - \bar{x}_v)$ for all $k \in V$; if no such $P_{l_k}^{1,h,k}$ or $P_{l_k}^{0,h,k}$ exists, we set $|\hat{P}_{l_k}^{1,h,k}| - 2\bar{x}(\hat{P}_{l_k}^{1,h,k}) = +\infty$ or $|\hat{P}_{l_k}^{0,h,k}| - 2\bar{x}(\hat{P}_{l_k}^{0,h,k}) = +\infty$ respectively. (Alternatively, we could remove that edge temporarily from the graph, but as we are mainly concerned with walks shorter than $2 - 2\bar{x}_h$ and $2\bar{x}_h$ respectively, those involving arcs with $|\hat{P}_{l_k}^{1,h,k}| - 2\bar{x}(\hat{P}_{l_k}^{1,h,k}) = +\infty$ or $|\hat{P}_{l_k}^{0,h,k}| - 2\bar{x}(\hat{P}_{l_k}^{0,h,k}) = +\infty$ would never be used.)

Our approach towards polynomial separation is now the following:

- A most violated wheel is by previous results determined by its hub, rim and the spoke ends on the rim (the spokes themselves do not matter, since by Lemma 5 they are just shortest walks of appropriate parity).
- Hence the task of finding a most violated wheel for a given hub reduces to determine a rim and the spoke ends.
- Hence we will not have to worry about the spokewalks as long as we manage to distribute their weight along the edges of the rim so that they sum up correctly.
- This requires, that in the original graph an edge will have a weight depending on its function (spoke end-to-spike end, spoke end-to-internal, internal-to-internal).
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- For a given rim-edge of a given wheel, we know of course ex-post what function it has; so first we will distribute the weights ex-post and later we have to see by a duplication procedure, how to distribute different weights to the same edge depending on the (ex-ante) unknown function it might take in a wheel.

Now we have to analyze the summand $-2x_h + |P_i| - 2x(P_i) + |P_{i+1}| - 2x(P_{i+1}) + 2\sum_{i=1}^{j-1} (1-x_i-x_{i+1})$ from the right handsides of equations (1) and (2) more carefully. First we do an example about how to distribute the weight to the edges of the rim of a wheel, then we prove the insight.

**Example 6.** For the plain nonsimple wheel $W(0;1;1;1 < 4 < 5)$ with $k = 1$ of Figure 3(a) the plain inequality $[I']_A$ would be:

$$1x_h + (x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_1) + (x_8 + x_9) \leq 3 + \frac{2}{2}.$$  

We have already form [I] and its fourfold is:

$$2 - 2x_0 \leq (-2x_0 + (0) + (2 - 2x_8 - 2x_9) + 2(1 - x_1 - x_2) + 2(1 - x_2 - x_3) + 2(1 - x_3 - x_4)) + \sum_{i=1}^{j-1} (1-x_i-x_{i+1})$$

With this representation, we have a way to distribute the total weight of the wheel according to the three lines on the right to the three rim walks 1 − 2 − 3 − 4 and 4 − 5 and 5 − 6 − 2 − 1, respectively. In Figure 3(b) we dropped the spokes.

The situation makes pretty clear that we want to associate ex-post the weight $(-2x_0 + (2 - 2x_8 - 2x_9) + (0) + 2(1 - x_4 - x_5))$ to the edge 4 − 5. For the walk 1 − 2 − 3 − 4 and the weight

$$(-2x_0 + (0) + (2 - 2x_8 - 2x_9) + 2(1 - x_1 - x_2) + 2(1 - x_2 - x_3) + 2(1 - x_3 - x_4))$$

it is less clear how to distribute the weight to the edges. Certainly, nonnegative edge-weights are preferable, as they permit faster shortest path algorithms. Another desideratum is, that the weights of internal edges of a rimwalk, should not depend on the previous and next spoke.

One way to distribute them is to assign to 1 − 2, 2 − 3, 3 − 4 the weights

$$2(1 - x_1 - x_2) - 2x_0 + (0)$$

$$2(1 - x_2 - x_3)$$

$$2(1 - x_3 - x_4) - 0x_0 + 2(1 - x_8 + x_9).$$

The second weight is clearly nonnegative if $x$ fulfills the edge inequalities, since $x_2 + x_3 \leq 1$ is of course the same as $1 - x_2 - x_3 \geq 0$. At first sight, it seems awkward, that the weights of the first and last edge are not symmetric regarding $x_0$ (the first edge has $-2x_0$ whereas the last one has $0x_0$). But the asymmetric distribution of $x_0$ has the advantage that the last edge weight is also nonnegative, if $x$ fulfills edge inequalities. Only the first edge might (actually, should) have negative weight, but we will see later how asymmetry makes the negative weights more tractable.
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Depending on whether we consider inequality $I_A$ or $I_B$ the $P$ would be $P^1$ or $P^0$. If $l_{j+1} - l_j = 1$ (i.e., there is a single edge between spoke ends) then we have

$$-2x_h + |\dot{P}_{h,l_j}| - 2x(\dot{P}_{h,l_j}) - |\dot{P}_{h,l_{j+1}}| + 2 \sum_{i=l_j}^{l_{j+1} - 1} (1 - x_i - x_{i+1})$$

$$= 2(1 - x_{l_j} - x_{l_{j+1}} - 2x_h + |\dot{P}_{h,l_j}| - 2x(\dot{P}_{h,l_j}) - 2x(\dot{P}_{h,l_{j+1}})$$

and going from spoke end $l_j$ to $l_{j+1}$ incurs a cost of $2(1 - x_{l_j} - x_{l_{j+1}}) - 2x_h + |\dot{P}_{h,l_j}| - 2x(\dot{P}_{h,l_j}) + |\dot{P}_{h,l_{j+1}}| + 2x(\dot{P}_{h,l_{j+1}})$ along the single edge. Notice that for $\bar{x} \in \text{CSTAB}(G)$ this weight is nonnegative.

Otherwise $l_{j+1} - l_j \geq 3$ (i.e., there are at least three edges between the spoke ends):

$$-2x_h + |\dot{P}_{h,l_j}| - 2x(\dot{P}_{h,l_j}) + |\dot{P}_{h,l_{j+1}}| - 2x(\dot{P}_{h,l_{j+1}}) + 2 \sum_{i=l_j}^{l_{j+1}-1} (1 - x_i - x_{i+1})$$

$$= 2(1 - x_{l_j} - x_{l_{j+1}} - x_h) + |\dot{P}_{h,l_j}| - 2x(\dot{P}_{h,l_j})$$

$$+ 2 \sum_{i=l_j}^{l_{j+1}-2} (1 - x_i - x_{i+1})$$

$$+ 2(1 - x_{l_{j+1}-1} - x_{l_{j+1}} - ox_h) + |\dot{P}_{h,l_{j+1}}| - 2x(\dot{P}_{h,l_{j+1}}).$$

This suggests to distribute the weight differently to the edges of $l_j - (l_j + 1) - \cdots - (l_{j+1} - 1) - l_{j+1}$. So here we would want an edge $\{l_j, l_{j+1}\}$ leaving the spoke end to contribute $(2 - 2x_h - 2x_{l_j} - 2x_{l_{j+1}}) + |\dot{P}_{h,l_j}| - 2x(\dot{P}_{h,l_j})$, the internal edges $\{i, i+1\}$ not incident with the spoke ends to contribute $2(1 - x_i - x_{i+1})$, and the final edge $\{l_{j+1} - 1, l_{j+1}\}$ to contribute $(2 - 2x_{l_{j+1}-1} - 2x_{l_{j+1}}) + |\dot{P}_{h,l_{j+1}}| - 2x(\dot{P}_{h,l_{j+1}})$. Notice that the second and third weight are positive, if $\bar{x}$ fulfills the edge inequalities, but the first weight could be negative.

Given, that we know now a way to distribute the weights to edges ex-post, we need a way to achieve the same ex-ante. Towards this, we will investigate the digraph $F$ from Figure 4 and show, that there is a nice map from the rim of a wheel to it. Then, to exploit it, we will extend the map to a map from wheels to cycles in the product of the graph with $F$.

Define the digraph $F$ by Figure 4 where undirected edges represent pairs of antiparallel arcs.

Theorem 7 (Homomorphism). Given a wheel $W$ with rim $v_1, v_2, \ldots, v_{2k'+1}$ There exists a “homomorphism” $\phi$ of $v_1, v_2, \ldots, v_{2k'+1}, v_{2k'+2}$ (where we treat $v_1$ and $v_{2k'+2}$ as different) to $F$ so that

1. for all $1 \leq i \leq 2k'$ the pair $(\phi(v_i), \phi(v_{i+1}))$ is either a forward arc in $F$ or one of the undirected edges,
2. for all spoke-ends $l_i$ holds $\phi(v_i) \in \{0, 3\}$
3. $\phi(v_1) = 0$ and $\phi(v_{2k'+2}) = 3$. 

3 SEPARATION OF $I'_A$ AND $I'_B$

(a) A plain nonsimple wheel $W(0; 1; 1 < 4 < 5)$ with $k = 1$; vertex 2 stems from an identification of two different vertices; the parallel edge from 1 to 2 is not necessary but will simplify notation later and serves as a reminder of the identification.

(b) Here the hub and the spokes are removed leaving only the rim of the wheel.

Figure 3: A plain nonsimple wheel $W(0; 1; 1 < 4 < 5)$ with $k = 1$ and its rim highlighted on the right.

4. $\phi(v) \in \{0, 3\}$ implies that $v$ is a spoke end, and

5. $|\phi^{-1}(0)| + |\phi^{-1}(3)| \geq 4$.

The same holds for even wheels.

Proof. Consider an odd $(2k' + 1, 2k + 1)$-wheel $W(h; k'; k; 1 = l_1 < l_2 < \ldots < l_{2k+1} \leq 2k' + 1; P_{1_1}^1, P_{2_1}^1, \ldots, P_{1_2k'+1}^1)$ of $G$ with rim $v_1, \ldots, v_{2k' + 1}$. Start with setting $\phi(v_1) = 0$; now, given some $\phi(v_j)$, we have to choose $\phi(v_{j+1})$:

Case 1: If $v_j$ is a spoke end (hence $\phi(v_j) \in \{0, 3\}$) and $v_{j+1}$ is another spoke end, then we set $\phi(v_{j+1}) = (\phi(v_j) + 3 \mod 6)$.

Case 2: If $v_j$ is a spoke end (hence $\phi(v_j) \in \{0, 3\}$) and $v_{j+1}$ is no spoke end, then we set $\phi(v_{j+1})$ equal to $\phi(v_j) + 1$.

Case 3: If $v_j$ and $v_{j+1}$ are both no spoke ends, that is, $\phi(v_j) \in \{1, 2, 4, 5\}$, then we set $\phi(v_{j+1})$ so that either $\{\phi(v_j), \phi(v_{j+1})\} = \{1, 2\}$ or $\{\phi(v_j), \phi(v_{j+1})\} = \{4, 5\}$.

Case 4: If finally $v_j$ is not a spoke end but $v_{j+1}$ is a spoke end, then we have to argue that $\phi(v_j)$ equals 2 or 5 since otherwise we can not reach in that step one of the nodes 3 or 0. So suppose $\phi(v_j) \in \{1, 2\}$, hence the most recently visited spoke end had second component 0. By assumption on the wheel the diwalk from that spoke end to $v_j$ is even. Therefore $\phi(v_j) \in \{1, 2\}$.

Figure 4: The digraph $F$ (undirected edges represent pairs antiparallel arcs).
Finally, as the number of nodes of the rim is odd, \( \phi(v_{2k+2}) = 3 \). (In this way, we have constructed a corresponding \( 0 \rightsquigarrow 3 \) diwalk in \( D \).) As every subdivided wheel has at least 3 spoke ends, the diwalk contains at least twice the vertices 0 and twice the 3.

The previous result demonstrates already a way to get from wheels to walks. To utilize this way further we define a product of graphs: The **categorical product** \( G_1 \times D_2 \) of a graph \( G_1 \) and a digraph \( D_2 \) is defined by \( V(G_1 \times D_2) = V(G_1) \times V(D_2) \) and \( A(G_1 \times D_2) = \{((u_1, u_2), (v_1, v_2)) : (u_1, v_1) \in E(G_1) \text{ and } (u_2, v_2) \in A(D_2)\} \). For \( G = (V, E) \) consider the digraph \( D := G \times F \), with \( F \) depicted in Figure 4. We want to embed the violated-odd/even-wheel-with-hub-finding-task into this graph. Interpret vertices of type \( V \times \{0, 3\} \) as vertices that correspond to spoke ends. Define for any given \( x \in Q^{V(F)} \) and any vertex \( h \in V \) the weighted digraph \( D_h := D \) where the arc \( e = ((u, i), (v, j)) \) has the following weight:

\[
w^1_e = \begin{cases} 
2(1 - x_u - x_v) - 2x_h & \text{if } \{i, j\} = \{0, 3\} \\
+|\hat{P}^{1}_{h,u}| - 2x(\hat{P}^{1}_{h,u}) + |\hat{P}^{1}_{h,v}| - 2x(\hat{P}^{1}_{h,v}) & \text{if } \{i, j\} = \{(3, 4), (0, 1)\} \\
2(1 - x_u - x_v) - 2x_h + |\hat{P}^{1}_{h,u}| - 2x(\hat{P}^{1}_{h,u}) & \text{if } \{i, j\} = \{(2, 3), (5, 0)\} \\
2(1 - x_u - x_v) - 0x_h + |\hat{P}^{1}_{h,v}| - 2x(\hat{P}^{1}_{h,v}) & \text{if } \{i, j\} = \{1, 2\} \\
2(1 - x_u - x_v) & \text{if } \{i, j\} = \{4, 5\} 
\end{cases}
\]

Notice that these weights pick up the idea of Example \( \boxed{3} \) generalize them and put them accordingly into the product-graph. We define another set of weights \( w^0 \) analogously in terms of \( P^0 \) for separation of \( I'_E \).

**Lemma 8.** For any diwalk \( U = (v_1, i_1) - (v_2, i_2) - \cdots - (v_q, i_q) \) in \( D_h \) with \( \{i_1, i_q\} = \{0, 3\} \neq i_2, \ldots, i_{q-1} \) with \( q \geq 2 \), and \( \hat{x} \in \text{ESTAB}(G) \) holds:

(a) \( q \) is even.

(b) \( w^1(U) := \sum_{j=1}^{q-1} w^1_{v_j,v_{j+1}} = -2x_h + |\hat{P}^{1}_{h,v_1}| - 2x(\hat{P}^{1}_{h,v_1}) + 2\sum_{k=1}^{q-1}(1 - x_{v_k} - x_{v_{k+1}}) + |\hat{P}^{1}_{h,v_q}| - 2x(\hat{P}^{1}_{h,v_q}) \) and the same for \( w^0(U) \) in terms of \( P^0 \).

(c) If \( \hat{x} \in \text{CSTAB}(G) \) then \( w^1(U) \geq 0 \leq w^0(U) \).

(d) If \( \hat{x} \in \text{CSTAB}(G) \) and \( v_1 = v_q \), then \( w^1(U) \geq 2 - 2x_h \) and \( w^0(U) \geq 2x_h \).

**Proof.** Clearly, there are only the two equivalent cases \( (i_1, i_q) = (0, 3) \) and \( (i_1, i_q) = (3, 0) \); so assume the first. Claim \( \boxed{a} \) follows immediately from the simple observation that \( D_h \) is bipartite (partition the vertices of \( D_h \) according to the parity of their second component) with \( (v_1, 0) \) and \( (v_q, 3) \) in different parts; let \( r = q/2 \).

The case of \( q = 2 \) of Claim \( \boxed{b} \) follows readily from the definition of the weights.
Otherwise \( q \geq 4 \):

\[
w^1(U) := \left( \sum_{j=1}^{q-1} w^1_{(v_j,i_j),(v_j+1,i_{j+1})} \right)
\]

\[
= \left( w^1_{(v_1,i_1),(v_2,i_2)} + \sum_{j=2}^{q-2} w^1_{(v_j,i_j),(v_j+1,i_{j+1})} + w^1_{(v_{q-1},i_{q-1}),(v_1,i_1)} \right)
\]

\[
= (2 - 2\bar{x}_h - 2\bar{x}_{v_1} - 2\bar{x}_{v_2}) + |\bar{P}_{h,v_1}^1| - 2\bar{x}(\bar{P}_{h,v_1}^1) + 2 \sum_{i=2}^{q-2} (1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}})
\]

\[
+ (2 - 2\bar{x}_{v_{q-1}} - 2\bar{x}_{v_q}) + |\bar{P}_{h,v_q}^1| - 2\bar{x}(\bar{P}_{h,v_q}^1);
\]

(3)

the last step follows from the definition of \( w \) and from the observation that \( i_2, \ldots, i_{q-1} \in \{1, 2\} \); the argument for \( w^0 \) is the same.

Claim \([\text{c}]\) is clearly valid, if there is no odd path from \( h \) to \( v_1 = v_q \) since then by definition either \( |\bar{P}_{h,v_q}^1| - 2\bar{x}(\bar{P}_{h,v_q}^1) = +\infty \) or \( |\bar{P}_{h,v_q}^1| - 2\bar{x}(\bar{P}_{h,v_q}^1) = +\infty \). So if these odd paths exist rearrange terms in \([\text{3}]\) to obtain

\[
w^1(U)
\]

\[
= -2\bar{x}_h + 2 \sum_{i=1}^{2r-1} (1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}}) + |\bar{P}_{h,v_1}^1| - 2\bar{x}(\bar{P}_{h,v_1}^1) + |\bar{P}_{h,v_q}^1| - 2\bar{x}(\bar{P}_{h,v_q}^1)
\]

\[
= -2\bar{x}_h + 2 \sum_{i=1}^{r} (1 - \bar{x}_{v_{2i-1}} - \bar{x}_{v_{2i}}) + 2 \sum_{i=1}^{r-1} (1 - \bar{x}_{v_{2i}} - \bar{x}_{v_{2i+1}})
\]

\[
+ |\bar{P}_{h,v_1}^1| - 2\bar{x}(\bar{P}_{h,v_1}^1) + |\bar{P}_{h,v_q}^1| - 2\bar{x}(\bar{P}_{h,v_q}^1)
\]

\[
= 2 \left( r + \frac{|\bar{P}_{h,v_1}^1| + |\bar{P}_{h,v_q}^1|}{2} \right) - \bar{x}_h - \sum_{i=1}^{2r} \bar{x}_{v_i} - \bar{x}(\bar{P}_{h,v_1}^1) - \bar{x}(\bar{P}_{h,v_q}^1)
\]

\[
+ 2 \sum_{i=1}^{r-1} (1 - \bar{x}_{v_{2i}} - \bar{x}_{v_{2i+1}})
\]

The first term’s nonnegativity is equivalent to \( \bar{x}_h + \sum_{i=1}^{2r} \bar{x}_{v_i} + \bar{x}(\bar{P}_{h,v_1}^1) + \bar{x}(\bar{P}_{h,v_q}^1) \leq r + \frac{|\bar{P}_{h,v_1}^1| + |\bar{P}_{h,v_q}^1|}{2} \), which is just an odd cycle inequality for \( h, \bar{P}_{h,v_1}^1, v_1, \ldots, v_{2r}, \bar{P}_{h,v_q}^1, h \) that is fulfilled by assumption. The nonnegativity of the second term is implied by a bunch of edge constraints of type \( \bar{x}_{v_{2i}} + \bar{x}_{v_{2i+1}} \leq 1 \). The argument for \( w^0 \) is the same.

Claim \([\text{d}]\) for \( w^1 \): As there is no arc between \((v_1,0)\) and \((v_1,3)\) in \( D_h \) (since \( G \) has
no loops), it follows that \( q \geq 3 \). Hence the claim is equivalent to

\[
2 - 2\bar{x}_h \leq w^1(U)
\]

\[
= -2\bar{x}_h + 2 \sum_{i=1}^{2r-1} (1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}})
+ |P^1_{h,v_1}| - 2\bar{x}(P^1_{h,v_1}) + |P^1_{h,v_q}| - 2\bar{x}(P^1_{h,v_q})
\]

\[
2 \leq 2 \sum_{i=1}^{2r-1} (1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}})
+ (|P^1_{h,v_1}| - 2\bar{x}(P^1_{h,v_1})) + (|P^1_{h,v_q}| - 2\bar{x}(P^1_{h,v_q})).
\]

As nonnegativity of the last two terms in parentheses is implied by the edge inequalities, it suffices to prove \( 2 \leq 2 \sum_{i=1}^{2r-1} (1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}}) \), which by \( v_1 = v_{2r} \) reduces to

\[
4 \left( \sum_{i=1}^{2r-1} \bar{x}_{v_i} \leq r - 1 \right).
\]

This is equivalent to the odd \( C_{2r-1} \) inequality through \( v_1, \ldots, v_{2r-1} \) being fulfilled, which is a consequence of \( \bar{x} \in \text{CSTAB}(G) \).

For \( w^0 \) the claim is equivalent to

\[
2 \leq 2 \sum_{i=1}^{2r-1} (1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}})
+ (|P^0_{h,v_1}| + 1 - 2\bar{x}_h - 2\bar{x}(P^0_{h,v_1})) + (|P^0_{h,v_q}| + 1 - 2\bar{x}_h - 2\bar{x}(P^0_{h,v_q})).
\]

As nonnegativity of the last two terms in parentheses is implied by the edge inequalities with \( v_1 = v_{2r} \), it suffices to prove: \( 4 \left( \sum_{i=1}^{2r-1} \bar{x}_{v_i} \leq r - 1 \right) \). This is equivalent to the odd \( C_{2r-1} \) inequality through \( v_1, \ldots, v_{2r-1} \) being fulfilled which is a consequence of \( \bar{x} \in \text{CSTAB}(G) \).

**Lemma 9.** Every (nonsimple) odd wheel \( W(h;k';k;1 = l_1 < l_2 < \cdots < l_{2k+1} \leq 2k' + 1; P^1_{l_1}, P^1_{l_2}, \ldots, P^1_{l_{2k'+1}}) \) of \( G \) using shortest spoke paths corresponds to a \((v_1, 0) \rightsquigarrow (v_3, 3)\) diwalk \( U \) (containing at least 3 vertices with second component 0 or 3) in \( D_h \) of the same finite weight with respect to \( w \) and \( \bar{x} \in \text{CSTAB}(G) \) and vice-versa. An analogous statement holds for (nonsimple) even \((2k' + 1, 2k + 1)\)-wheels \( W(h;k';k;1 = l_1 < l_2 < \cdots < l_{2k+1} \leq 2k' + 1; P^0_{l_1}, P^0_{l_2}, \ldots, P^0_{l_{2k'+1}}) \) and shortest \((v_1, 0) \rightsquigarrow (v_3, 3)\) diwalks (containing at least 3 vertices with second component 0 or 3) with respect to \( w^0 \) too.

**Proof.** Two properties of \((v, 0) \rightsquigarrow (v, 3)\) walks \( U = (v_1 = v, i_1 = 0) \cdots \cdots (v_p = v, i_p = 3) \) in \( D_h \) are:

1. Since the graph underlying \( D_h \) is bipartite (as \( F \) is) and the sequence \( i_1, i_2, \ldots, i_p \) is alternating between odd and even starting with even and ending with odd, \( p \) has to be even.
2. As the second component of the start vertex of every arc in $D_h$ from a vertex with second component from the set $\{1, 2, 3\}$ to one with second component from $\{4, 5, 0\}$ equals 3 and any arc from $\{4, 5, 0\}$ to $\{1, 2, 3\}$ starts in 0 we see that $|\{j \in \{1, \ldots, p\}: i_j \in \{0, 3\}\}$ is even.

We start with the reverse direction and let $k = (|\{j \in \{1, \ldots, p\}: i_j \in \{0, 3\}\}| - 2)/2$ and $k' = (p - 2)/2$. By the assumption that the diwalk $U$ meets at least three vertices with second component 0 or 3 it follows that $k \geq 1$; trivially holds $k' \geq k$. Now let $l_1, \ldots, l_{2k+2}$ be the indices $l$ corresponding to $i_l \in \{0, 3\}$ in the order in which they are encountered by the diwalk $U$.

We want to argue that $v_1, \ldots, v_{2k'+1}$, together with the spoke ends $v_1, \ldots, v_{2k+1}$, form an odd $(2k'+1, 2k+1)$-wheel. Consider the subdiwalk from $v_{l_i}$ to $v_{l_{i+1}}$; as both vertices have second component from $\{0, 3\}$, we know by Lemma [8(a)] that the subwalk from $(v_{l_i}, i_{l_i})$ to $(v_{l_{i+1}}, i_{l_{i+1}})$ in $U$ is odd, that is $i_{l_{i+1}} - i_{l_i}$ is odd. As the diwalk has finite weight and therefore $|P_{h,v_i}| - 2\bar{x}(P_{h,v_i}) \neq +\infty$ the odd walks $P_{h,v_i}$ exist and can be used as spokes. So we have associated a (nonsimple) odd wheel with a diwalk.

The forward direction is covered already by Theorem 7. Take the map $\phi$ from there and set $i_{j} = \phi(v_{j})$.

Application of Lemma [8(b)] on the terms in parentheses in

$$\sum_{j=1}^{2k+1} \left(-2\bar{x}_h + |P_{l_i}^k| - 2\bar{x}(P_{l_i}^k) + |P_{l_{i+1}}^k| - 2\bar{x}(P_{l_{i+1}}^k) + 2 \sum_{i=l_i}^{l_{i+1} - 1} (1 - \bar{x}_i - \bar{x}_{i+1})\right)$$

yields immediately that the length of the diwalk and the weight of the wheel are equal.

Hence, to find a violated odd (even) wheel with hub $h$ and initial spoke end $v_1$ we have to find a shortest odd (even) wheel with hub $h$ and initial spoke end $v_1$; if its weight is less than $2(1 - \bar{x}_h)$ (or $2\bar{x}_h$) then it is violated. Otherwise there is no violated odd (even) wheel with hub $h$ and initial spoke end $v_1$. To reduce it to a shortest path problem in $D_h$ where arc-weights $w^1, w^0$ can be negative we have to ensure that $D_h$ contains no negative diycle.

**Lemma 10.** Given some $\bar{x} \in \text{CSTAB}(G)$ and a vertex $h$ of $G$ then $D_h$ contains no negative diycle with respect to $w^1$ and none with respect to $w^0$.

**Proof.** Consider a diycle $(v_1, i_1) - \cdots - (v_p = v_1, i_p = i_1)$ of $D_h$; first of all, its length has to be even (as $F$ is bipartite), that is, $p$ is odd; so let $k = (p - 1)/2$. If all second components belong to $\{1, 2\}$ (or to $\{4, 5\}$), all involved edge weights are nonnegative, since they have the form $2(1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}})$ which is nonnegative as the edge inequalities are fulfilled. If the second components do not all belong to either $\{1, 2\}$ or $\{4, 5\}$, then there is a vertex with second component from $\{0, 3\}$. Without loss of generality, we may assume that $i_1 = 0$ and can now as before determine a sequence $l_1 = 1 < l_2 < \cdots < l_{2k+1}$ of indices of vertices on the walk that have second component 0 or 3; again, the differences between consecutive $l_i$ are odd. The weights of the subdiwalks from $(v_{l_i}, i_{l_i})$ to $(v_{l_{i+1}}, i_{l_{i+1}})$ are nonnegative by Lemma [8(c)]. Hence the claim follows for $w^1$ and analogously for $w^0$.

The task to find a shortest subdivided wheel with hub $h$ and initial spoke end $v_1$ can be solved by finding a shortest $(v_1, 0) \rightsquigarrow (v_1, 3)$ diwalk encountering at least two more
vertices with second component in $\{0,3\}$ in $D_h$ with respect to weights $w^1$ as defined for Lemma [5] and comparing that length to $2 - 2\bar{x}_h$. By Lemma [6,4] if the shortest $(v_1, 0) \leadsto (v_1, 3)$ diwalk in $D_h$ has only two vertices with second component in $\{0,3\}$, then its weight is at least $2 - 2\bar{x}_h$, thereby certifying that no violated wheel with hub $h$ and start $v$ exists. This yields the next result.

**Corollary 11.** Given $G$ and $\bar{x} \in \text{CSTAB}(G)$ then there is a violated odd wheel-inequality with hub $h$ starting in $v$ iff $D_h$ contains a $(v, 0) \leadsto (v, 3)$ diwalk of length less than $2 - 2\bar{x}_h$ with respect to $w^1$. There is a violated even wheel-inequality with hub $h$ starting in $v$ iff $D_h$ contains a $(v, 0) \leadsto (v, 3)$ diwalk of length less than $2\bar{x}_h$ with respect to $w^0$.

Finally we consider the complexity of the entire separation algorithm for a graph $G$ with $n$ vertices and $m$ edges. It is easy to see that for every hub $h \in V$ first of all we have to compute the odd $P^*_{h}^i$-walks and even $P^N_{h}^i$-walks having at least one arc each as candidates for spokes with one call to Dijkstra in $O(nm + n^2 \log n)$ we check whether there is a $(v, 0) \leadsto (v, 3)$ diwalk of length less than $2 - 2\bar{x}_h$ or less than $2\bar{x}_h$, respectively. As there are $n$ hubs we achieve an overall running time of $O(n^2m + n^3 \log n)$. Thus we have proved:

**Theorem 12.** The separation problems given $\bar{x} \in \text{CSTAB}(G)$ for $\text{WA$\text{STAB}$}(G)$ and $\text{W$_B$STAB}(G)$ can be solved in time $O(n^2m + n^3 \log n)$.

This is an improvement over the $O(n^4)$-bound for the algorithm by Cheng and Cunningham [8, 9] for the separation problem if the involved graphs have $m = O(n^3)$ for some $\beta < 2$ (i.e. their average degree is bounded by some constant or $O(n^\epsilon)$ for some $\epsilon < 1$). We found for the conflict graphs occurring during a branch-and-cut solution of the ORLIB-problems of Beasley with odd-cycle- and clique-cuts, that $m < 17n$, hinting that even $\beta = 1$ is plausible. Moreover, the case $\beta = 2$ is only of little interest for LP-based approaches, as then the graph is almost complete, which provides stable-set-instances that are amendable to more combinatorial approaches.

### 4. Speed-Ups for Practise

In practise, one often wants to separate only a *promising subset* of the spoke end-hub-pairs and not over all combination. Suppose for some heuristic, we want to search for a violated inequality for $N$ hubs and a total of $M$ spoke end-hub-pairs. For each hub, it is necessary to compute a potential by using an $O(mn)$ shortest path algorithm. Further, each of the $M$ spoke end-hub-pairs require a $O(m + n \log n)$ application of the Dijkstra algorithm. Taken together this requires $O(M(m + n \log n) + N(mn))$. Therefore for the initial version of the algorithm, it is always faster to check several spoke end-hub-pairs for the same hub than for different hubs. We will decompose the problem slightly different, so that we achieve a time of $O((M + N)(m + n \log n))$ so that during the execution the most promising spoke end-hub-pairs can be choosen without a substantial performance degradation.

Given some digraph $D = (V, A, w)$ with arbitrary (possibly negative) arcweights but without negative cycles, a *potential* is a $p \in \mathbb{Q}^V$ that fulfills

$$w_{uv} \geq p_v - p_u \quad \forall uv \in A.$$
During Johnson’s algorithm, first a potential of the digraph is computed that is used to define new nonnegative weights $w'_{uv} := w_{uv} - p_u + p_v$. It turns out, that shortest paths with respect to $w$ are shortest paths with respect to $w'$ and vice versa. However, due to the special structure of our digraph $D_h$, it is possible to compute a potential not only in $O(mn)$ (Bellman-Ford) but already in $O(m + n \log n)$ (Dijkstra).

The standard way would be to add an artificial vertex $s$ to $D_h$ to obtain digraph $D'_h$ where arcs of length zero are going from $s$ to any vertex of $D'_h$. Now the shortest path distances from $s$ in $D'_h$ yield (when restricted to $D_h$) a potential. Given that negative arcs but not negative cycles might be present, this can be solved by the Bellman-Ford algorithm.

To avoid the $O(mn)$ Bellman-Ford algorithm, the idea is to fix some distances taking care of all negative arcs and then running a slight variation of Dijkstra’s algorithm.

**Lemma 13.** For given $\bar{x} \in \text{CSTAB}(G)$, and $h \in G$ the vertices $(v, 0)$ and $(v, 3)$ are at distance 0 from $s$ in $D'_h$ with respect to $w^1$ (or $w^0$).

**Proof.** Because of $\bar{x} \in \text{CSTAB}(G)$ the graph $D'_h$ contains no negative cycles. Let $(v, i)$ be the vertex closest to $s$ among all $(w, 0), (w, 3)$ (and if there are multiple ones, then one closest with respect of the number of edges is choosen). If the interior of the shortest path from $s$ to $(v, i)$ would contain another vertex from $(w, 0), (w, 3)$, we could choose the last such vertex $(u, j)$ on that path before $(v, i)$ (hence $j = i + 3 \mod 6$) and let $U$ be the path from $(u, j)$ to $(v, i)$; by [6] of Lemma 8 we have $w^1(U) \geq 0$; hence $(u, j)$ would be fewer edges away from $s$ than $(v, i)$ and its distance would not be larger either, contradicting the choice of $(v, i)$. Hence, on the path from $s$ to $(v, i)$ are no vertices of type $(w, 0), (w, 3)$ and hence no negative edges, which ensures that the distance from $s$ to $(v, i)$ is $\geq 0$. \qed

Now in the Dijkstra algorithm on $D'_h$, we label all vertices $(v, 0)$ and $(v, 3)$ with 0 and make these labels final. For each arc $((v, 0/3), (u, 1/4))$ we update the distance label of $(u, 1/4)$ if the arc permits to reduce its distance. It is pivotal to notice, that the arc $((v, 0/3), (u, 1/4))$ can not bring another improvement afterward, since its source $(v, 0/3)$ got already a final distance label. After this update, the arcs $((v, 0/3), (u, 1/4))$ can be removed from $D'_h$. The resulting network has no longer any negative arcs, hence Dijkstra’s algorithm can finish the computation of the distance labels. So we have proved:

**Lemma 14.** Given $\bar{x} \in \text{CSTAB}(G)$ and $h \in G$, a potential for $D_h$ can be computed in time $O(m + n \log n)$.

This yields the promised result:

**Theorem 15.** The separation problem given $\bar{x} \in \text{CSTAB}(G)$ for $W_A \text{STAB}(G)$ and $W_B \text{STAB}(G)$ can be solved in time by $O(n^2)$ applications of the fast Dijkstra algorithm (each requiring $O(m + n \log n)$); of these, $O(n)$ Dijkstra invocations are used to precompute potentials and then $O(n^2)$ invocations are used to search violated wheels.

5. Conclusion

The categorical product of the original graph with a small gadget presented a new way to keep track of the different parity requirements when separating general wheel
inequalities. The advantage of our approach over the approach by Cheng and Cunningham \cite{8,9}, is that our approach preserves the sparsity of the graph, while their approach involves an auxiliary graph that is dense. So when facing problems on sparse graphs, the approach via categorical graph product might beneficial.

**References**