Faster Separation of 1-Wheel Inequalities by Graph Products

Sven de Vries[a,1]

[a]*FB IV – Mathematik, Universität Trier, 54286 Trier, Germany*
phone: +49 651 201 3476; fax: +49 651 201 3479; devries@uni-trier.de

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Abstract

Using graph products we present an $O(|V|^2|E| + |V|^3 \log |V|)$ separation algorithm for the nonsimple 1-wheel inequalities by Cheng and Cunningham (1997) of the stable set polytope, which is faster than their $O(|V|^4)$ algorithm.

There are two ingredients for our algorithm. The main improvement stems from a reduction of separation problem to multiple shortest path problems in an auxillary graph having only $6|V|$ vertices and $9|E|$ arcs, thereby preserving low sparsity. Then Johnson's algorithm can be applied exploiting that preserved sparsity of the original graph which is maintained in the auxillary graph.

In contrast, Cheng and Cunningham's auxillary graph is by construction dense, $|E| = O(|V|^2)$, so application of Johnson's algorithm provides no large improvement.

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URL: `http://www.mathematik.uni-trier.de/~devries` (Sven de Vries)

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Many important application problems contain large subproblems of the following *binary packing* type:

$$\begin{aligned} \max\ & c^\top x \\ \text{s.t.}\ & Ax \le b \\ & x \in \{0,1\}^n \end{aligned} \tag{BPP}$$

where (A, b) is a matrix of nonnegative integers. In the process of solving (BPP) by a branch-and-cut algorithm, it is for the cutting-part helpful to associate with it Padberg's *conflict/intersection graph,* see [1, 2]. Let $V = \{1, \ldots, n\}$ and any pair of vertices i, j is adjacent if for the columns a^i and a^j holds that $a^i + a^j \not\leq b$. We denote the resulting conflict graph with $G_{A,b}$. Now any integral and feasible x^* is also a feasible solution for the stable set polytope of $G_{A,b}$. Hence the stable set polytope of $G_{A,b}$ provides an integral relaxation of (BPP).

1. Introduction

Let $G = (V, E)$ be a simple connected graph with $|V| = n \geq 2$ and $|E| = m$. A subset of V is called *stable* if it does not contain adjacent vertices of G. The *incidence vector* of a set $N \subseteq V$ is $\chi^N \in \{0,1\}^V$ such that $\chi^N_v = 1$ if $v \in N$ and otherwise $\chi^N_v = 0$. The *stable set polytope* of G, denoted by STAB(G), is the convex hull of incidence vectors of stable sets of G. Some well-known valid inequalities for STAB(G) include the *trivial inequalities* ($x_v \geq 0$ for $v \in V$), the *odd cycle inequalities* ($\sum_{v \in C} x_v \leq k$ where C is the vertex-set of an odd cycle of length $2k + 1$), and the *clique inequalities* ($\sum_{v \in K} x_v \leq 1$ where K induces a clique). A clique inequality is called *edge inequality* if the clique has just two vertices. Let $\text{ESTAB}(G) := \{x \in [0,1]^V : x_u + x_v \leq 1\ \forall uv \in E\}$ and $\text{CSTAB}(G) := \{x \in \text{ESTAB}(G) : x \text{ fulfills the odd cycle inequalities}\}$.

The *separation problem* for a class $\mathcal{C}$ of valid inequalities for a class of polytopes P is: Given $x^* \in P$, does x^* violate one of the inequalities in $\mathcal{C}$? If the answer is *yes*, exhibit such an inequality. Solving this problem is important to use the inequalities in a branch-and-cut approach for maximizing a linear function over some general integer program or STAB(G). (See, for example, Barahona et al. [3] and Nemhauser and Sigismondi [4].) Furthermore, if the separation problem for $\mathcal{C}$ is solvable in polynomial time, then the linear optimization problem over $\mathcal{C}$ can be solved in polynomial time, see Grötschel et al. [5]. The separation problem for $\mathcal{C} = \{\text{trivial and edge inequalities}\}$ can obviously be solved in $O(m)$ time. If x^* satisfies the trivial and edge inequalities, then one can decide whether x^* violates an odd cycle inequality in polynomial time, as was first observed by Grötschel and Pulleyblank [6], Grötschel et al. [5]. Odd cycle inequalities can be separated by n applications of the fast Dijkstra algorithm by Fredman and Tarjan [7] in time $O(nm + n^2 \log n)$. Hence the separation problem for the trivial, the edge, and the odd cycle inequalities can be solved in the same time.

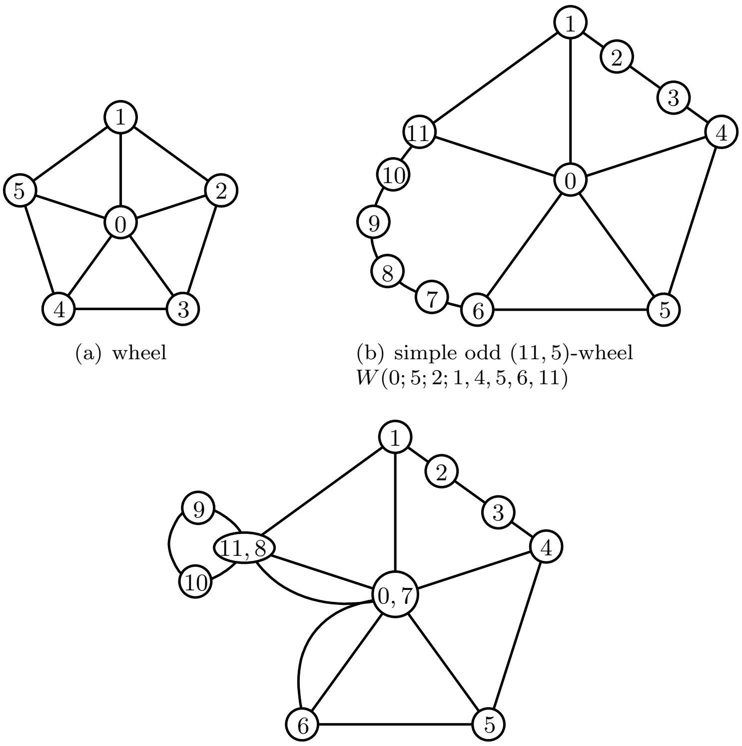
Cheng and Cunningham [8, 9] describe a way to separate the 1-wheel inequalities in time $O(n^4)$. They achieve this, by reducing the separation problem to multiple shortest path problems in dense graphs on $O(n)$ vertices.

In the present study we reduce the complexity of the separation problem of 1-wheels down to $O(n^2 m + n^3 \log n)$. As stable set problems often originate from the conflict graphs of more general integer programs, and as these conflict graphs tend to be sparse, this faster algorithm is important for practical applications.

In contrast to Cheng and Cunningham's approach we construct a new auxillary graph that is the categorical product of the original graph and a gadget on just 6 vertices; separation boils down to shortest path problems in this auxillary graph solved by Johnson's algorithm. As the runtime of Johnson's algorithm depends on the number of edges of the auxillary graph, which for our construction is just a constant multiple of the original number of edges, our approach is able to exploit sparsity of the original graph.

Plain application of Johnson's algorithm to Cheng and Cunningham's auxillary graph can not achieve a comparable speed-up, as their auxillary graph is dense by construction.

Our approach will be to consider a least-weight wheel problem and its reduction to a shortest path problem in a that product graph. Then we show that the separation problem reduces to the least-weight wheel problem.



(a) wheel

(b) simple odd $(11, 5)$-wheel $W(0; 5; 2; 1, 4, 5, 6, 11)$

(c) nonsimple odd wheel obtained by identifying vertices $0, 7$ and $8, 11$

Figure 1: From a wheel to a simple odd $(11, 5)$-wheel $W(0; 5; 2; 1, 4, 5, 6, 11)$ to a nonsimple odd wheel obtained by identifying vertices $0, 7$ and $8, 11$.

2. Wheels

Cheng and Cunningham [8, 9] consider a wheel with $(2k + 1)$ vertices and hub h (for an example of a wheel on 5 vertices and hub, see Figure 1(a)) and its subdivisions, see Figure 1(b). Let $1, \dots, 2k'+1$ be the rim where the *spoke ends* are l_1 up to l_{2k+1}, ordered so that $1 = l_1 < l_2 < \cdots < l_{2k+1} \le 2k'+1$. Denote the *spoke paths* connecting h to some l_i by P_{l_i} and their subpaths that exclude both ends by $\mathring{P}_{l_i}$. With $|\mathring{P}_{l_i}|$ we denote the number of vertices in $\mathring{P}_{l_i}$. Let the interior of the spoke paths be pairwise disjoint and let the interior of the spoke paths be disjoint to the rim. A *wheel* has to fulfill additionally the condition that the cycles $h, \mathring{P}_{l_i}, l_i, l_i{+}1, \dots, l_{i+1}, \mathring{P}_{l_{i+1}}, h$ are odd for $i = 1, 2, \dots, 2k{+}1$; for a complete specification we denote it by $W(h; k'; k; l_1, l_2, \dots, l_{2k+1}; P_{l_1}, P_{l_2}, \dots, P_{l_{2k+1}})$.

Let $\mathcal{E}$ be the set of the l_i for which the paths P_{l_i} have an *even* number of edges, and let $\mathcal{O}$ be the set of remaining spoke ends. Cheng and Cunningham [8, 9] show that the inequalities

$$kx_h + \sum_{i=1}^{2k'+1} x_i + \sum_{i \in \mathcal{E}} x_i + \sum_{i=1}^{2k+1} x(\mathring{P}_{l_i}) \le k' + \frac{|\mathcal{E}| + \sum_{i=1}^{2k+1} |\mathring{P}_{l_i}|}{2} \qquad (I_{\mathcal{A}})$$

$$(k+1)x_h + \sum_{i=1}^{2k'+1} x_i + \sum_{i \in \mathcal{O}} x_i + \sum_{i=1}^{2k+1} x(\mathring{P}_{l_i}) \le k' + \frac{|\mathcal{O}| + 1 + \sum_{i=1}^{2k+1} |\mathring{P}_{l_i}|}{2} \qquad (I_{\mathcal{B}})$$

are valid and they give sufficent conditions for them to induce facets. (Here we use $x(\mathring{P})$ for a walk $P = v_0 - \cdots - v_{k+1}$ as a shorthand for $\sum_{i=1}^{k} x_{v_i}$.)

Proposition 1 ([8, Prop. 2.2])**.** *Let $\sum_{i=1}^{n} a_{v_i} x_{v_i} \le a_0$ be a valid inequality for STAB(G) and let v_1 and v_2 be two* nonadjacent *vertices of G. If H is obtained from G by identifying v_1 and v_2 to a single vertex $v_{1,2}$, then $(a_{v_1} + a_{v_2})x_{v_{1,2}} + \sum_{i=3}^{n} a_{v_i} x_{v_i} \le a_0$ is valid for STAB(H).*

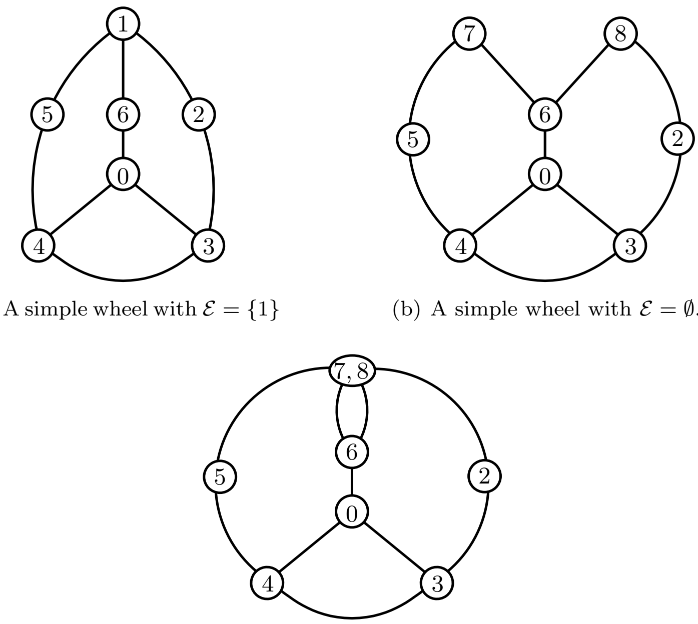
Therefore, when speaking of *general* or *nonsimple* wheels we will permit the identification of nonadjacent vertices where the coefficient of the new vertex is the sum of the coefficients of the identified vertices. The next example motivates how to avoid $\mathcal{E}$ for $I_{\mathcal{A}}$ and $\mathcal{O}$ for $I_{\mathcal{B}}$ by going nonsimple.

Example 2. *Consider the simple wheel of Figure 2(a) with $\mathcal{E} = \{1\}$ and its $I_{\mathcal{A}}$ inequality $1x_0 + \sum_{i=1}^{6} x_i + x_1 = 1x_0 + \sum_{i=2}^{6} x_i + 2x_1 \le 3$. The simple wheel in Figure 2(b) has the $I_{\mathcal{A}}$ inequality $x_0 + \sum_{i=2}^{8} x_i \le 3$ with $\mathcal{E} = \emptyset$. Now, by identifying the two vertices 7 and 8, we obtain the nonsimple wheel of Figure 2(c) and the inequality according to Lemma 1 is $x_0 + \sum_{i=2}^{6} x_i + 2x_{7,8} \le 3$ which is the same (when relabeling the vertex $\{7, 8\}$ by 1) as the original inequality that involved $\mathcal{E} \ne \emptyset$.*

More formally we obtain:

Lemma 3. *For a wheel and its nonsimple $I_{\mathcal{A}}$-inequality we can assume without loss of generality $\mathcal{E} = \emptyset$.*

Proof. Given a wheel with $v_\ell \in \mathcal{E}$ and its $I_{\mathcal{A}}$-inequality $\mathcal{I}$. Now consider the wheel resulting from contracting the edge of P_ℓ incident with v_ℓ and then subdividing the two rim edges incident with v_ℓ once (the resulting vertices are called v_i and v_j). Notice that another wheel results, $|\mathcal{E}|$ decreases by one, $|\mathring{P}_\ell|$ decreases by one and k' increases by one; so all together, we obtain a new wheel with $\mathcal{E}' = \mathcal{E} \setminus \{v_\ell\}$. The right hand side of the new



(a) A simple wheel with $\mathcal{E} = \{1\}$

(b) A simple wheel with $\mathcal{E} = \emptyset$.

(c) A nonsimple wheel with $\mathcal{E} = \emptyset$ and spoke ends $3, 4, 6$.

Figure 2: Getting away with $\mathcal{E} = \emptyset$ for $I_{\mathcal{A}}$.

$I_{\mathcal{A}}$-inequality $\mathcal{I}'$ equals the right hand side of $\mathcal{I}$. Now if the two new vertices v_i, v_j are identified (hence the coefficient of $v_{i,j}$ becomes two) then the inequality $\mathcal{I}$ results, now represented as a nonsimple wheel $I_{\mathcal{A}}$-inequality $\mathcal{I}''$ with $|\mathcal{E}''| < |\mathcal{E}|$. □

Since Cheng and Cunningham [9, Thm. 4.2] proved that for simple wheels the inequality $I_{\mathcal{B}}$ can induce facets only if the odd spokes have length at least 3, it is natural to restrict ourselves to these.

Lemma 4. *For a wheel without odd spokes of length < 3 its $I_{\mathcal{B}}$ inequality is representable by another $I_{\mathcal{B}}$-inequality with $\mathcal{O} = \emptyset$.*

As the proof is analogous to that of Lemma 3 we omit the repetition. In the sequel we will assume $\mathcal{E} = \emptyset$ for $(I_{\mathcal{A}})$ and $\mathcal{O} = \emptyset$ for $(I_{\mathcal{B}})$ permitting more concise notation. For $(I_{\mathcal{A}})$ we consider those simple wheels $W(h; k'; k; l_1, l_2, \ldots, l_{2k+1}; P_{l_1}, P_{l_2}, \ldots, P_{l_{2k+1}})$ with $\mathcal{E} = \emptyset$ and call them *plain odd* $(2k'+1, 2k+1)$*-wheels* or shorter *plain odd wheels*. Clearly they fulfill $l_{j+1} - l_j$ is odd for $j = 1, \ldots, 2k$ and that all spoke paths are odd. Similarly, for $(I_{\mathcal{B}})$ we consider those simple wheels with $\mathcal{O} = \emptyset$ and call them *plain even* $(2k'+1, 2k+1)$*-wheels*. Clearly they fulfill $l_{j+1} - l_j$ is odd for $j = 1, \ldots, 2k$ and that all spoke paths are even. So we obtain for an odd wheel $W(h; k'; k; 1 = l_1 < l_2 < \cdots < l_{2k+1} \le 2k'+1; P_{l_1}, P_{l_2}, \ldots, P_{l_{2k+1}})$ the following form of inequality $(I_{\mathcal{A}})$:

$$kx_h + \sum_{i=1}^{2k'+1} x_i + \sum_{i=1}^{2k+1} x(\mathring{P}_{l_i}) \le k' + \frac{\sum_{i=1}^{2k+1} |\mathring{P}_{l_i}|}{2}, \qquad (I'_{\mathcal{A}})$$

and for an even wheel the inequality ($I_{\mathcal{B}}$) simplifies to

$$(k+1)x_h + \sum_{i=1}^{2k'+1} x_i + \sum_{i=1}^{2k+1} x(\mathring{P}_{l_i}) \le k' + \frac{1+\sum_{i=1}^{2k+1}|\mathring{P}_{l_i}|}{2}. \qquad (I'_{\mathcal{B}})$$

Because of Proposition 1, we will focus on general (that is, possibly nonsimple) plain odd/even $(2k'+1, 2k+1)$-wheels. Define

$$\begin{aligned} \mathrm{W}_{\mathcal{A}}\mathrm{STAB}(G) &:= \{x \in \mathrm{CSTAB}(G) \colon x \text{ fulfills all plain nonsimple } I'_{\mathcal{A}}\text{-inequalities}\} \\ \mathrm{W}_{\mathcal{B}}\mathrm{STAB}(G) &:= \{x \in \mathrm{CSTAB}(G) \colon x \text{ fulfills all plain nonsimple } I'_{\mathcal{B}}\text{-inequalities}\}. \end{aligned}$$

3. Separation of $I'_{\mathcal{A}}$ and $I'_{\mathcal{B}}$

Consider the following form of the four-fold of ($I'_{\mathcal{A}}$) for an odd $(2k'+1, 2k+1)$-wheel $W(h; k'; k; 1 = l_1 < l_2 < \cdots < l_{2k+1} \le 2k'+1; P_{l_1}, P_{l_2}, \ldots, P_{l_{2k+1}})$:

$$2 - 2x_h + (4k+2)x_h + 4\sum_{i=1}^{2k'+1} x_i + 4\sum_{i=1}^{2k+1} x(\mathring{P}_{l_i}) \le (4k'+2) + 2\sum_{i=1}^{2k+1} |\mathring{P}_{l_i}|$$

Here and henceforth we identify the indices of spoke ends modulo $2k+1$ so that index $2k+2$ is identified with 1, index $2k+3$ is identified with 2 etc; reshuffling yields:

$$2 - 2x_h \le \sum_{j=1}^{2k+1}\Big(-2x_h + \big(|\mathring{P}_{l_j}| - 2x(\mathring{P}_{l_j})\big) + \big(|\mathring{P}_{l_{j+1}}| - 2x(\mathring{P}_{l_{j+1}})\big) + 2\sum_{i=l_j}^{l_{j+1}-1}(1 - x_i - x_{i+1})\Big). \quad (1)$$

Similarly, one obtains for an even wheel and ($I'_{\mathcal{B}}$):

$$2x_h \le \sum_{j=1}^{2k+1}\Big(-2x_h + \big(|\mathring{P}_{l_j}| - 2x(\mathring{P}_{l_j})\big) + \big(|\mathring{P}_{l_{j+1}}| - 2x(\mathring{P}_{l_{j+1}})\big) + 2\sum_{i=l_j}^{l_{j+1}-1}(1 - x_i - x_{i+1})\Big). \quad (2)$$

Call the right hand sides of equations (1) and (2) the *weights* of the odd/even wheel with respect to x. Notice that except for the fact that in (1) the P-paths are odd and in (2) they are even, the functional form of the weight is the same. Now we prove for given $\bar{x}, h$, rim and spoke ends a property of the spoke walks P_{l_j} in a *most violated* odd/even wheel inequality.

Lemma 5 (Theorem 3.5 [9])**.** *For a given graph G and $\bar{x} \in ESTAB(G)$ fix a hub h, an odd cycle $v_1, v_2, \ldots, v_{2k'+1}$, and spoke ends $1 = l_1 < l_2 < \cdots < l_{2k+1} \le 2k'+1$. Among* *all odd wheels with this hub and rim and these spoke ends consider one (with some P_{l_j}) such that*

$$2 - 2\bar{x}_h \leq \sum_{j=1}^{2k+1} \Bigg(-2\bar{x}_h + \big(|\mathring{P}_{l_j}| - 2\bar{x}(\mathring{P}_{l_j})\big) + \big(|\mathring{P}_{l_{j+1}}| - 2\bar{x}(\mathring{P}_{l_{j+1}})\big) + 2 \sum_{i=l_j}^{l_{j+1}-1} (1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}}) \Bigg) \qquad (I'_{\mathcal{A}})$$

is most violated, that is the right hand side is minimal. Then P_{l_j} is a shortest odd *walk from h to v_{l_j} with respect to the edge weights $1 - \bar{x}_u - \bar{x}_v$, which are nonnegative because of $\bar{x} \in CSTAB(G)$.*

Similarly for a most violated inequality $I'_{\mathcal{B}}$ for fixed hub and spoke ends it follows that P_{l_j} is a shortest even walk (among all even walks with at least 2 edges) from h to v_{l_j} with respect to the edge weights $(1 - \bar{x}_u - \bar{x}_v)$.

Proof. Consider the odd wheel $W(h; k'; k; 1 = l_1 < l_2 < \cdots < l_{2k+1} \leq 2k'+1; P_{l_1}, P_{l_2}, \ldots, P_{l_{2k+1}})$ and suppose P'_{l_j} is shorter than P_{l_j}, that is $\sum_{uv \in E(P'_{l_j})}(1 - \bar{x}_u - \bar{x}_v) < \sum_{uv \in E(P_{l_j})}(1 - \bar{x}_u - \bar{x}_v)$ or equivalently

$$2 + |\mathring{P}'_{l_j}| - 2\bar{x}(\mathring{P}'_{l_j}) - \bar{x}_h - \bar{x}_{v_{l_j}} < 2 + |\mathring{P}_{l_j}| - 2\bar{x}(\mathring{P}_{l_j}) - \bar{x}_h - \bar{x}_{v_{l_j}}.$$

This implies $|\mathring{P}'_{l_j}| - 2\bar{x}(\mathring{P}'_{l_j}) < |\mathring{P}_{l_j}| - 2\bar{x}(\mathring{P}_{l_j})$. But now the odd wheel W' that results from W by replacing P_{l_j} by P'_{l_j} has less weight than the supposedly minimum weight wheel W and therefore is more violated than W. Contradiction! The argument for $(I'_{\mathcal{B}})$ is analogous. □

From now on we assume for some fixed $\bar{x} \in$ ESTAB(G) that we have computed shortest odd walks $P^1_{h,k}$ and shortest even walks $P^0_{h,k}$ (the latter having at least two edges) with respect to edge weights $(1 - \bar{x}_u - \bar{x}_v)$ for all $k \in V$; if no such $P^1_{h,k}$ or $P^0_{h,k}$ exists, we set $|\mathring{P}^1_{h,k}| - 2x(\mathring{P}^1_{h,k}) = +\infty$ or $|\mathring{P}^0_{h,k}| - 2x(\mathring{P}^0_{h,k}) = +\infty$ respectively. (Alternatively, we could remove that edge temporarily from the graph, but as we are mainly concerned with walks shorter than $2 - 2\bar{x}_h$ and $2\bar{x}_h$ respectively, those involving arcs with $|\mathring{P}^1_{h,k}| - 2x(\mathring{P}^1_{h,k}) = +\infty$ or $|\mathring{P}^0_{h,k}| - 2x(\mathring{P}^0_{h,k}) = +\infty$ would never be used.)

Our *approach* towards polynomial separation is now the following:

- A most violated wheel is by previous results determined by its hub, rim and the spoke ends on the rim (the spokes themselves do not matter, since by Lemma 5, they are just shortest walks of appropriate parity).
- Hence the task of finding a most violated wheel for a given hub reduces to determine a rim and the spoke ends.
- Hence we will not have to worry about the spokewalks as long as we manage to *distribute* their weight along the edges of the rim so that they sum up correctly.
- This requires, that in the original graph an edge will have a weight depending on its function (spoke end-to-spoke end, spoke end-to-internal, internal-to-internal).

- For a given rim-edge of a given wheel, we know of course ex-post what function it has; so first we will distribute the weights ex-post and later we have to see by a duplication procedure, how to distribute different weights to the same edge depending on the (ex-ante) unkown function it might take in a wheel.

Now we have to analyze the summand $-2x_h + |\mathring{P}_{l_j}| - 2x(\mathring{P}_{l_j}) + |\mathring{P}_{l_{j+1}}| - 2x(\mathring{P}_{l_{j+1}}) + 2\sum_{i=l_j}^{l_{j+1}-1}(1-x_i-x_{i+1})$ from the right handsides of equations (1) and (2) more carefully. First we do an example about how to distribute the weight to the edges of the rim of a wheel, then we prove the insight.

Example 6. *For the plain nonsimple wheel $W(0;1;1;1<4<5)$ with $k=1$ of Figure 3(a) the plain inequality $(I'_{\mathcal{A}})$ would be:*

$$1x_h + (x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_1) + (x_8 + x_9) \leq 3 + \frac{2}{2}.$$

We have already form (1) and its fourfold is::

$$\begin{aligned}2 - 2x_0 \leq &(-2x_0 + (0) + (2 - 2x_8 - 2x_9) + 2(1 - x_1 - x_2) + 2(1 - x_2 - x_3) + 2(1 - x_3 - x_4)) \\ &+ (-2x_0 + (2 - 2x_8 - 2x_9) + (0) + 2(1 - x_4 - x_5)) \\ &+ (-2x_0 + (0) + (0) + 2(1 - x_5 - x_6) + 2(1 - x_6 - x_2) + 2(1 - x_2 - x_1))\end{aligned}$$

With this representation, we have a way to distribute the total weight of the wheel according to the three lines on the right to the three rim walks $1-2-3-4$ and $4-5$ and $5-6-2-1$, respectively. In Figure 3(b) we dropped the spokes.

The situation makes pretty clear that we want to associate ex-post the weight $(-2x_0 + (2 - 2x_8 - 2x_9) + (0) + 2(1 - x_4 - x_5))$ to the edge $4-5$. For the walk $1-2-3-4$ and the weight

$$(-2x_0 + (0) + (2 - 2x_8 - 2x_9) + 2(1 - x_1 - x_2) + 2(1 - x_2 - x_3) + 2(1 - x_3 - x_4))$$

it is less clear how to distribute the weight to the edges. Certainly, nonnegative edge-weights are preferable, as they permit faster shortest path algorithms. Another desideratum is, that the weights of internal edges of a rimwalk, should not depend on the previous and next spoke.

One way to distribute them is to assign to $1-2, 2-3, 3-4$ the weights

$$\begin{aligned}&2(1 - x_1 - x_2) - 2x_0 + (0) \\ &2(1 - x_2 - x_3) \\ &2(1 - x_3 - x_4) - 0x_0 + 2(1 - x_8 + x_9).\end{aligned}$$

The second weight is clearly nonnegative if x fulfills the edge inequalities, since $x_2+x_3 \leq 1$ is of course the same as $1-x_2-x_3 \geq 0$. At first sight, it seems awkward, that the weights of the first and last edge are not symmetric regarding x_0 (the first edge has $-2x_0$ whereas the last one has $-0x_0$). But the asymmetric distribution of x_0 has the advantage that the last edge weight is also nonnegative, if x fulfills edge inequalities. Only the first edge might (actually, should) have negative weight, but we will see later how asymmetry makes the negative weights more tractable.

Depending on whether we consider inequality ($I'_{\mathcal{A}}$) or ($I'_{\mathcal{B}}$) the $\mathring{P}$ would be $\mathring{P}^1$ or $\mathring{P}^0$. If $l_{j+1} - l_j = 1$ (i.e., there is a single edge between spoke ends) then we have

$$\begin{aligned}&-2x_h + |\mathring{P}_{h,l_j}| - 2x(\mathring{P}_{h,l_j}) + |\mathring{P}_{h,l_{j+1}}| - 2x(\mathring{P}_{h,l_{j+1}}) + 2\sum_{i=l_j}^{l_{j+1}-1}(1 - x_i - x_{i+1})\\ &\qquad = 2(1 - x_{l_j} - x_{l_{j+1}}) - 2x_h + |\mathring{P}_{h,l_j}| - 2x(\mathring{P}_{h,l_j}) + |\mathring{P}_{h,l_{j+1}}| - 2x(\mathring{P}_{h,l_{j+1}})\end{aligned}$$

and going from spoke end l_j to l_{j+1} incurs a cost of $2(1 - x_{l_j} - x_{l_{j+1}}) - 2x_h + |\mathring{P}_{h,l_j}| - 2x(\mathring{P}_{h,l_j}) + |\mathring{P}_{h,l_{j+1}}| - 2x(\mathring{P}_{h,l_{j+1}})$ along the single edge. Notice that for $\bar{x} \in \mathrm{CSTAB}(G)$ this weight is nonnegative.

Otherwise $l_{j+1} - l_j \geq 3$ (i.e., there are at least three edges between the spoke ends):

$$\begin{aligned}&-2x_h + |\mathring{P}_{h,l_j}| - 2x(\mathring{P}_{h,l_j}) + |\mathring{P}_{h,l_{j+1}}| - 2x(\mathring{P}_{h,l_{j+1}}) + 2\sum_{i=l_j}^{l_{j+1}-1}(1 - x_i - x_{i+1})\\ =&2(1 - x_{l_j} - x_{l_j+1} - x_h) + |\mathring{P}_{h,l_j}| - 2x(\mathring{P}_{h,l_j})\\ &+ 2\sum_{i=l_j+1}^{l_{j+1}-2}(1 - x_i - x_{i+1})\\ &+ 2(1 - x_{l_{j+1}-1} - x_{l_{j+1}} - 0x_h) + |\mathring{P}_{h,l_{j+1}}| - 2x(\mathring{P}_{h,l_{j+1}}).\end{aligned}$$

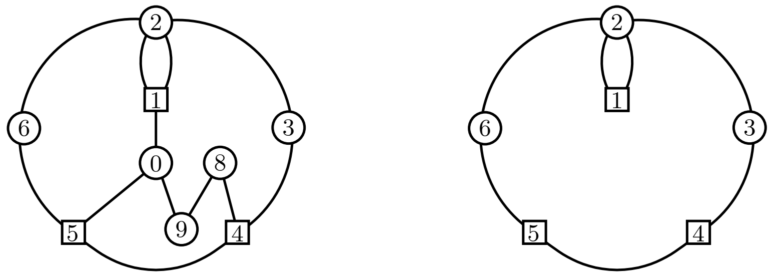
This suggests to distribute the weight differently to the edges of $l_j - (l_j + 1) - \cdots - (l_{j+1} - 1) - l_{j+1}$. So here we would want an edge $\{l_j, l_j + 1\}$ leaving the spoke endto contribute $(2 - 2x_h - 2x_{l_j} - 2x_{l_j+1}) + |\mathring{P}_{h,l_j}| - 2x(\mathring{P}_{h,l_j})$, the internal edges $\{i, i + 1\}$ not incident with the spoke ends to contribute $2(1 - x_i - x_{i+1})$, and the final edge $\{l_{j+1} - 1, l_{j+1}\}$ to contribute $(2 - 2x_{l_{j+1}-1} - 2x_{l_{j+1}}) + |\mathring{P}_{h,l_{j+1}}| - 2x(\mathring{P}_{h,l_{j+1}})$. Notice that the second and third weight are positive, if $\bar{x}$ fulfills the edge inequalities, but the first weight *could be negative.*

Given, that we know now a way to distribute the weights to edges ex-post, we need to find a way to achieve the same ex-ante. Towards this, we will investigate the digraph F from Figure 4, and show, that there is a nice map from the rim of a wheel to it. Then, to exploit it, we will extend the map to a map from wheels to cycles in the product of the graph with F.

Define the digraph F by Figure 4, where undirected edges represent pairs of antiparallel arcs.

Theorem 7 (Homomorphism)**.** *Given a wheel W with rim $v_1, v_2, \ldots, v_{2k'+1}$ There exists a "homomorphism" ϕ of $v_1, v_2, \ldots, v_{2k'+1}, v_{2k'+2}$ (where we treat v_1 and $v_{2k'+2}$ as different) to F so that*

1. *for all $1 \leq i \leq 2k' + 1$ the pair $(\phi(v_i), \phi(v_{i+1}))$ is either a forward arc in F or one of the undirected edges,*
2. *for all spoke-ends l_i holds $\phi(v_{l_i}) \in \{0, 3\}$*
3. *$\phi(v_1) = 0$ and $\phi(v_{2k'+2}) = 3$,*



(a) A plain nonsimple wheel $W(0;1;1;1 < 4 < 5)$ with $k = 1$; vertex 2 stems from an identification of two different vertices; the parallel edge from 1 to 2 is not necessary but will simplify notation later and serves as a reminder of the identification.

(b) Here the hub and the spokes are removed leaving only the rim of the wheel.

Figure 3: A plain nonsimple wheel $W(0;1;1;1 < 4 < 5)$ with $k = 1$ and its rim highlighted on the right.

4. $\phi(v) \in \{0,3\}$ implies that v is a spoke end, and

5. $|\phi^{-1}(0)| + |\phi^{-1}(3)| \geq 4$.

The same holds for even wheels.

Proof. Consider an odd $(2k'+1, 2k+1)$-wheel $W(h;k';k;1 = l_1 < l_2 < \cdots < l_{2k+1} \leq 2k'+1; P^1_{l_1}, P^1_{l_2}, \ldots, P^1_{l_{2k'+1}})$ of G with rim $v_1, \ldots, v_{2k'+1}$. Start with setting $\phi(v_1) = 0$; now, given some $\phi(v_j)$, we have to choose $\phi(v_{j+1})$:

Case 1: If v_j is a spoke end (hence $\phi(v_j) \in \{0,3\}$) and v_{j+1} is another spoke end, then we set $\phi(v_{j+1}) = (\phi(v_j) + 3 \bmod 6)$.

Case 2: If v_j is a spoke end (hence $\phi(v_j) \in \{0,3\}$) and v_{j+1} is no spoke end, then we set $\phi(v_{j+1})$ equal to $\phi(v_j) + 1$.

Case 3: If v_j and v_{j+1} are both no spoke ends, that is, $\phi(v_j) \in \{1,2,4,5\}$, then we set $\phi(v_{j+1})$ so that either $\{\phi(v_j), \phi(v_{j+1})\} = \{1,2\}$ or $\{\phi(v_j), \phi(v_{j+1})\} = \{4,5\}$.

Case 4: If finally v_j is not a spoke end but v_{j+1} is a spoke end, then we have to argue that $\phi(v_j)$ equals 2 or 5 since otherwise we can not reach in that step one of the nodes 3 or 0. So suppose $\phi(v_j) \in \{1,2\}$, hence the most recently visited spoke end had second component 0. By assumption on the wheel the diwalk from that spoke end to v_j is even. Therefore $\phi(v_j)$ has to be 2.

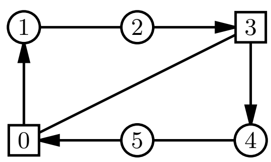


Figure 4: The digraph F (undirected edges represent pairs antiparallel arcs).

Finally, as the number of nodes of the rim is odd, $\phi(v_{2k'+2}) = 3$. (In this way, we have constructed a corresponding $0 \rightsquigarrow 3$ diwalk in D.) As every subdivided wheel has at least 3 spoke ends, the diwalk contains at least twice the vertices 0 and twice the 3. □

The previous result demonstrates already a way to get from wheels to walks. To utilize this way further we define a product of graphs: The *categorical product* $G_1 \cdot D_2$ of a graph G_1 and a digraph D_2 is defined by $V(G_1 \cdot D_2) = V(G_1) \times V(D_2)$ and $A(G_1 \cdot D_2) = \{((u_1, u_2), (v_1, v_2)) \colon (u_1, v_1) \in E(G_1) \text{ and } (u_2, v_2) \in A(D_2)\}$. For $G = (V, E)$ consider the digraph $D := G \cdot F$, with F depicted in Figure 4. We want to embed the violated-odd/even-wheel-with-hub-h-finding-task into this graph. Interpret vertices of type $V \times \{0, 3\}$ as vertices that correspond to spoke ends. Define for any given $\bar{x} \in \mathbb{Q}^{V(G)}$ and any vertex $h \in V$ the weighted digraph $D_h := D$ where the arc $e = ((u, i), (v, j))$ has the following weight:

$$w^1_e = \begin{cases} 2(1 - \bar{x}_u - \bar{x}_v) - 2\bar{x}_h \\ \qquad + |\mathring{P}^1_{h,u}| - 2\bar{x}(\mathring{P}^1_{h,u}) + |\mathring{P}^1_{h,v}| - 2\bar{x}(\mathring{P}^1_{h,v}) & \text{if } \{i, j\} = \{0, 3\} \\ 2(1 - \bar{x}_u - \bar{x}_v) - 2\bar{x}_h + |\mathring{P}^1_{h,u}| - 2\bar{x}(\mathring{P}^1_{h,u}) & \text{if } (i, j) \in \{(3, 4), (0, 1)\} \\ 2(1 - \bar{x}_u - \bar{x}_v) - 0\bar{x}_h + |\mathring{P}^1_{h,v}| - 2\bar{x}(\mathring{P}^1_{h,v}) & \text{if } (i, j) \in \{(2, 3), (5, 0)\} \\ 2(1 - \bar{x}_u - \bar{x}_v). & \text{if } \{i, j\} \in \{\{1, 2\}, \{4, 5\}\} \end{cases}$$

Notice that these weights pick up the idea of Example 6, generalize them and put them accordingly into the product-graph. We define another set of weights w^0 analogously in terms of $\mathring{P}^0$ for separation of $I'_{\mathcal{B}}$.

Lemma 8. *For any diwalk $U = (v_1, i_1) - (v_2, i_2) - \cdots - (v_q, i_q)$ in D_h with $\{i_1, i_q\} = \{0, 3\} \not\ni i_2, \ldots, i_{q-1}$, with $q \geq 2$, and $\bar{x} \in ESTAB(G)$ holds:*

(a) q is even.

(b) $w^1(U) := \sum_{j=1}^{q-1} w^1_{v_j, v_{j+1}} = -2\bar{x}_h + |\mathring{P}^1_{h,v_1}| - 2\bar{x}(\mathring{P}^1_{h,v_1}) + 2\sum_{i=1}^{q-1}(1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}}) + |\mathring{P}^1_{h,v_q}| - 2\bar{x}(\mathring{P}^1_{h,v_q})$ and the same for $w^0(U)$ in terms of $\mathring{P}^0$.

(c) If $\bar{x} \in CSTAB(G)$ then $w^1(U) \geq 0 \leq w^0(U)$.

(d) If $\bar{x} \in CSTAB(G)$ and $v_1 = v_q$, then $w^1(U) \geq 2 - 2\bar{x}_h$ and $w^0(U) \geq 2\bar{x}_h$.

Proof. Clearly, there are only the two equivalent cases $(i_1, i_q) = (0, 3)$ and $(i_1, i_q) = (3, 0)$; so assume the first. Claim (a) follows immediately from the simple observation that D_h is bipartite (partition the vertices of D_h according to the parity of their second component) with $(v_1, 0)$ and $(v_q, 3)$ in different parts; let $r = q/2$.

The case of $q = 2$ of Claim (b) follows readily from the definition of the weights.

Otherwise $q \geq 4$:

$$\begin{aligned} w^1(U) &:= \left(\sum_{j=1}^{q-1} w^1_{(v_j,i_j),(v_{j+1},i_{j+1})} \right) \\ &= \left(w^1_{(v_1,i_1),(v_2,i_2)} + \sum_{j=2}^{q-2} w^1_{(v_j,i_j),(v_{j+1},i_{j+1})} + w^1_{(v_{q-1},i_{q-1}),(v_q,i_q)} \right) \\ &= \quad (2 - 2\bar{x}_h - 2\bar{x}_{v_1} - 2\bar{x}_{v_2}) + |\mathring{P}^1_{h,v_1}| - 2\bar{x}(\mathring{P}^1_{h,v_1}) + 2\sum_{i=2}^{q-2}(1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}}) \\ &\quad + (2 - 2\bar{x}_{v_{q-1}} - 2\bar{x}_{v_q}) + |\mathring{P}^1_{h,v_q}| - 2\bar{x}(\mathring{P}^1_{h,v_q}); \end{aligned} \tag{3}$$

the last step follows from the definition of w and from the observation that $i_2, \ldots, i_{q-1} \in \{1, 2\}$; the argument for w^0 is the same.

Claim (c) is clearly valid, if there is no odd path from h to v_1 or h to v_q since then by definition either $|\mathring{P}^1_{h,v_1}| - 2\bar{x}(\mathring{P}^1_{h,v_1}) = +\infty$ or $|\mathring{P}^1_{h,v_q}| - 2\bar{x}(\mathring{P}^1_{h,v_q}) = +\infty$. So if these odd paths exist rearrange terms in (3) to obtain

$$\begin{aligned} &w^1(U) \\ =& -2\bar{x}_h + 2\sum_{i=1}^{2r-1}(1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}}) + |\mathring{P}^1_{h,v_1}| - 2\bar{x}(\mathring{P}^1_{h,v_1}) + |\mathring{P}^1_{h,v_q}| - 2\bar{x}(\mathring{P}^1_{h,v_q}) \\ =& -2\bar{x}_h + 2\sum_{i=1}^{r}(1 - \bar{x}_{v_{2i-1}} - \bar{x}_{v_{2i}}) + 2\sum_{i=1}^{r-1}(1 - \bar{x}_{v_{2i}} - \bar{x}_{v_{2i+1}}) \\ &+ |\mathring{P}^1_{h,v_1}| - 2\bar{x}(\mathring{P}^1_{h,v_1}) + |\mathring{P}^1_{h,v_q}| - 2\bar{x}(\mathring{P}^1_{h,v_q}) \\ =& 2\left(r + \frac{|\mathring{P}^1_{h,v_1}| + |\mathring{P}^1_{h,v_q}|}{2} - \bar{x}_h - \sum_{i=1}^{2r}\bar{x}_{v_i} - \bar{x}(\mathring{P}^1_{h,v_1}) - \bar{x}(\mathring{P}^1_{h,v_q}) \right) \\ &+ 2\sum_{i=1}^{r-1}(1 - \bar{x}_{v_{2i}} - \bar{x}_{v_{2i+1}}) \end{aligned}$$

The first term's nonnegativity is equivalent to $\bar{x}_h + \sum_{i=1}^{2r}\bar{x}_{v_i} + \bar{x}(\mathring{P}^1_{h,v_1}) + \bar{x}(\mathring{P}^1_{h,v_q}) \leq r + \frac{|\mathring{P}^1_{h,v_1}|+|\mathring{P}^1_{h,v_q}|}{2}$ which is just an odd cycle inequality for $h, \mathring{P}^1_{h,v_1}, v_1, \ldots, v_{2r}, \mathring{P}^1_{h,v_{2r}}, h$ that is fulfilled by assumption. The nonnegativity of the second term is implied by a bunch of edge constraints of type $\bar{x}_{v_{2i}} + \bar{x}_{v_{2i+1}} \leq 1$. The argument for w^0 is the same.

Claim (d) for w^1 : As there is no arc between $(v_1, 0)$ and $(v_1, 3)$ in D_h (since G has no loops), it follows that $q \geq 3$. Hence the claim is equivalent to

$$\begin{aligned} 2 - 2\bar{x}_h \leq & w^1(U) \\ & = -2\bar{x}_h + 2 \sum_{i=1}^{2r-1} (1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}}) \\ & \quad + |\mathring{P}^1_{h,v_1}| - 2\bar{x}(\mathring{P}^1_{h,v_1}) + |\mathring{P}^1_{h,v_q}| - 2\bar{x}(\mathring{P}^1_{h,v_q}) \\ 2 \leq & 2 \sum_{i=1}^{2r-1} (1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}}) \\ & \quad + (|\mathring{P}^1_{h,v_1}| - 2\bar{x}(\mathring{P}^1_{h,v_1})) + (|\mathring{P}^1_{h,v_q}| - 2\bar{x}(\mathring{P}^1_{h,v_q})). \end{aligned}$$

As nonnegativity of the last two terms in parentheses is implied by the edge inequalities, it suffices to prove $2 \leq 2 \sum_{i=1}^{2r-1} (1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}})$, which by $v_1 = v_{2r}$ reduces to

$$4 \left(\sum_{i=1}^{2r-1} \bar{x}_{v_i} \leq r - 1 \right).$$

This is equivalent to the odd C_{2r-1} inequality through $v_1, \ldots, v_{2r-1}$ being fulfilled, which is a consequence of $\bar{x} \in \mathrm{CSTAB}(G)$.

For w^0 the claim is equivalent to

$$\begin{aligned} 2 \leq & 2 \sum_{i=1}^{2r-1} (1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}}) \\ & + (|\mathring{P}^0_{h,v_1}| + 1 - 2\bar{x}_h - 2\bar{x}(\mathring{P}^0_{h,v_1})) + (|\mathring{P}^0_{h,v_q}| + 1 - 2\bar{x}_h - 2\bar{x}(\mathring{P}^0_{h,v_q})). \end{aligned}$$

As nonnegativity of the last two terms in parentheses is implied by the edge inequalities with $v_1 = v_{2r}$ it suffices to prove: $4 \left(\sum_{i=1}^{2r-1} \bar{x}_{v_i} \leq r - 1 \right)$. This is equivalent to the odd C_{2r-1} inequality through $v_1, \ldots, v_{2r-1}$ being fulfilled which is a consequence of $\bar{x} \in \mathrm{CSTAB}(G)$. □

Lemma 9. *Every (nonsimple) odd wheel* $W(h; k'; k; 1 = l_1 < l_2 < \cdots < l_{2k+1} \leq 2k' + 1; P^1_{l_1}, P^1_{l_2}, \ldots, P^1_{l_{2k'+1}})$ *of* G *using shortest spoke paths corresponds to a* $(v_1, 0) \rightsquigarrow (v_1, 3)$ *diwalk* U *(containing at least* 3 *vertices with second component* 0 *or* 3*) in* D_h *of the same finite weight with respect to* w *and* $\bar{x} \in CSTAB(G)$ *and vice-versa. An analogous statement holds for (nonsimple) even* $(2k' + 1, 2k + 1)$*-wheels* $W(h; k'; k; 1 = l_1 < l_2 < \cdots < l_{2k+1} \leq 2k' + 1; P^0_{l_1}, P^0_{l_2}, \ldots, P^0_{l_{2k'+1}})$ *and shortest* $(v_1, 0) \rightsquigarrow (v_1, 3)$ *diwalks (containing at least* 3 *vertices with second component* 0 *or* 3*) with respect to* w^0 *too.*

Proof. Two properties of $(v, 0) \rightsquigarrow (v, 3)$ walks $U = (v_1 = v, i_1 = 0) - \cdots - (v_p = v, i_p = 3)$ in D_h are:

1. Since the graph underlying D_h is bipartite (as F is) and the sequence $i_1, i_2, \ldots, i_p$ is alternating beetween odd and even starting with even and ending with odd, p has to be even.

2. As the second component of the start vertex of every arc in D_h from a vertex with second component from the set $\{1,2,3\}$ to one with second component from $\{4,5,0\}$ equals 3 and any arc from $\{4,5,0\}$ to $\{1,2,3\}$ starts in 0 we see that $|\{j \in \{1,\dots,p\}: i_j \in \{0,3\}\}|$ is even.

We start with the reverse direction and let $k = (|\{j \in \{1,\dots,p\}: i_j \in \{0,3\}\}| - 2)/2$ and $k' = (p-2)/2$. By the assumption that the diwalk U meets at least three vertices with second component 0 or 3 it follows that $k \geq 1$; trivially holds $k' \geq k$. Now let $l_1,\dots,l_{2k+2}$ be the indices l corresponding to $i_l \in \{0,3\}$ in the order in which they are encountered by the diwalk U.

We want to argue that $v_1,\dots,v_{2k'+1}$ together with the spoke ends $v_{l_1},\dots,v_{l_{2k+1}}$ form an odd $(2k'+1, 2k+1)$-wheel. Consider the subdiwalk from v_{l_j} to $v_{l_{j+1}}$; as both vertices have second component $\{0,3\}$, we know by Lemma 8(a) that the subwalk from (v_{l_j}, i_{l_j}) to $(v_{l_{j+1}}, i_{l_{j+1}})$ in U is odd, that is $i_{l_{j+1}} - i_{l_j}$ is odd. As the diwalk has finite weight and therefore $|\mathring{P}^1_{h,v_{l_j}}| - 2x(\mathring{P}^1_{h,v_{l_j}}) \neq +\infty$ the odd walks $P^1_{h,v_{l_j}}$ exist and can be used as spokes. So we have associated a (nonsimple) odd wheel with a diwalk.

The forward direction is covered already by Theorem 7. Take the map ϕ from there and set $i_j = \phi(v_j)$.

Application of Lemma 8(b) on the terms in parentheses in

$$\sum_{j=1}^{2k+1} \left(-2\bar{x}_h + |\mathring{P}^1_{l_j}| - 2\bar{x}(\mathring{P}^1_{l_j}) + |\mathring{P}^1_{l_{j+1}}| - 2\bar{x}(\mathring{P}^1_{l_{j+1}}) + 2 \sum_{i=l_j}^{l_{j+1}-1} (1 - \bar{x}_i - \bar{x}_{i+1}) \right)$$

yields immediately that the length of the diwalk and the weight of the wheel are equal. □

Hence, to find a *violated* odd (even) wheel with hub h and initial spoke end v_1 we have to find a *shortest* odd (even) wheel with hub h and initial spoke end v_1; if its weight is less than $2(1-\bar{x}_h)$ (or $2\bar{x}_h$) then it is violated. Otherwise there is no violated odd (even) wheel with hub h and initial spoke end v_1. To reduce it to a shortest path problem in D_h where arc-weights w^1, w^0 can be negative we have to ensure that D_h contains no negative dicycle.

Lemma 10. *Given some $\bar{x} \in CSTAB(G)$ and a vertex h of G then D_h contains no negative dicycle with respect to w^1 and none with respect to w^0.*

Proof. Consider a dicycle $(v_1, i_1) - \cdots - (v_p = v_1, i_p = i_1)$ of D_h; first of all, its length has to be even (as F is bipartite), that is, p is odd; so let $k = (p-1)/2$. If all second components belong to $\{1,2\}$ (or to $\{4,5\}$), all involved edge weights are nonnegative, since they have the form $2(1 - \bar{x}_{v_i} - \bar{x}_{v_{i+1}})$ which is nonnegative as the edge inequalities are fulfilled. If the second components do not all belong to either $\{1,2\}$ or $\{4,5\}$, then there is a vertex with second component from $\{0,3\}$. Without loss of generality, we may assume that $i_1 = 0$ and can now as before determine a sequence $l_1 = 1 < l_2 < \cdots < l_{2k+1}$ of indices of vertices on the walk that have second component 0 or 3; again, the differences between consecutive l_j are odd. The weights of the subdiwalks from (v_{l_j}, i_{l_j}) to $(v_{l_{j+1}}, i_{l_{j+1}})$ are nonnegative by Lemma 8(c). Hence the claim follows for w^1 and analogously for w^0. □

The task to find a shortest subdivided wheel with hub h and initial spoke end v_1 can be solved by finding a shortest $(v_1, 0) \rightsquigarrow (v_1, 3)$ diwalk encountering at least two more vertices with second component in $\{0,3\}$ in D_h with respect to weights w^1 as defined for Lemma 8 and comparing that length to $2-2\bar{x}_h$. By Lemma 8(d), if the shortest $(v_1,0) \rightsquigarrow (v_1,3)$ diwalk in D_h has only two vertices with second component in $\{0,3\}$, then its weight is at least $2-2\bar{x}_h$, thereby certifying that no violated wheel with hub h and start v exists. This yields the next result.

Corollary 11. *Given G and $\bar{x} \in CSTAB(G)$ then there is a violated odd wheel-inequality with hub h starting in v iff D_h contains a $(v,0) \rightsquigarrow (v,3)$ diwalk of length less than $2-2\bar{x}_h$ with respect to w^1. There is a violated even wheel-inequality with hub h starting in v iff D_h contains a $(v,0) \rightsquigarrow (v,3)$ diwalk of length less than $2\bar{x}_h$ with respect to w^0.*

Finally we consider the complexity of the entire separation algorithm for a graph G with n vertices and m edges. It is easy to see that for every hub $h \in V$ first of all we have to compute the odd $P^1_{l_j}$-walks and even $P^0_{l_j}$-walks having at least one arc each as candidates for spokes with one call to Dijkstra in $O(m+n\log n)$ for $G \cdot K_2$ and for $G \cdot P_3$ (where P_3 is the path on 3 vertices), respectively. By applying Johnson's [1977] all-pairs-shortest-path algorithm to D_h in time $O(nm+n^2\log n)$ we check whether there is a $(v,0) \rightsquigarrow (v,3)$ diwalk of length less than $2-2\bar{x}_h$ or less than $2\bar{x}_h$, respectively. As there are n hubs we achieve an overall running time of $O(n^2m+n^3\log n)$. Thus we have proved:

Theorem 12. *The separation problems given $\bar{x} \in CSTAB(G)$ for $W_{\mathcal{A}}STAB(G)$ and $W_{\mathcal{B}}STAB(G)$ can be solved in time $O(n^2m+n^3\log n)$.*

This is an improvement over the $O(n^4)$-bound for the algorithm by Cheng and Cunningham [8, 9] for the separation problem if the involved graphs have $m = O(n^\beta)$ for some $\beta < 2$ (i.e. if their average degree is bounded by some constant or $O(n^\epsilon)$ for some $\epsilon < 1$). We found for the conflict graphs occurring during a branch-and-cut solution of the ORLIB-problems of Beasley with odd-cycle- and clique-cuts, that $m < 17n$, hinting that even $\beta = 1$ is plausible. Moreover, the case $\beta = 2$ is only of little interest for LP-based approaches, as then the graph is almost complete, which provides stable-set-instances that are amendable to more combinatorial approaches.

4. Speed-Ups for Practise

In practise, one often wants to separate only a *promising subset* of the spoke end-hub-pairs and not over all combination. Suppose for some heuristic, we want to search for a violated inequality for N hubs and a total of M spoke end-hub-pairs. For each hub, it is necessary to compute a potential by using an $O(mn)$ shortest path algorithm. Further, each of the M spoke end-hub-pairs require a $O(m+n\log n)$ application of the Dijkstra algorithm. Taken together this requires $O(M(m+n\log n)+N(mn))$. Therefore for the initial version of the algorithm, it is always faster to check several spoke end-hub-pairs for the same hub than for different hubs. We will decompose the problem slightly different, so that we achieve a time of $O((M+N)(m+n\log n))$ so that during the execution the most promising spoke end-hub-pairs can be choosen without a substantial performance degradation.

Given some digraph $D=(V,A,w)$ with arbitrary (possibly negative) arcweights but without negative cycles, a *potential* is a $p \in \mathbb{Q}^V$ that fulfills

$$w_{uv} \geq p_v - p_u \quad \forall uv \in A.$$

During Johnson's algorithm, first a potential of the digraph is computed that is used to define new nonnegative weights $w'_{uv} := w_{uv} - p_v + p_u$. It turns out, that shortest paths with respect to w are shortest paths with respect to w' and vice versa. However, due to the special structure of our digraph D_h, it is possible to compute a potential not only in $O(mn)$ (Bellman-Ford) but already in $O(m + n \log n)$ (Dijkstra).

The standard way would be to add an artificial vertex s to D_h to obtain digraph D'_h where arcs of length zero are going from s to any vertex of D'_h. Now the shortest path distances from s in D'_h yield (when restricted to D_h) a potential. Given that negative arcs but not negative cycles might be present, this can be solved by the Bellman-Ford algorithm.

To avoid the $O(mn)$ Bellman-Ford algorithm, the idea is to fix some distances taking care of all negative arcs and then running a slight variation of Dijkstra's algorithm.

Lemma 13. *For given $\bar{x} \in CSTAB(G)$, and $h \in G$ the vertices $(v, 0)$ and $(v, 3)$ are at distance 0 from s in D'_h with respect to w^1 (or w^0).*

Proof. Because of $\bar{x} \in \text{CSTAB}(G)$ the graph D'_h contains no negative cycles. Let (v, i) be the vertex closest to s among all $(w, 0), (w, 3)$ (and if there are multiple ones, then one closest with respect of the number of edges is choosen). If the interior of the shortest path from s to (v, i) would contain another vertex from $(w, 0), (w, 3)$, we could choose the last such vertex (u, j) on that path before (v, i) (hence $j = i + 3 \mod 6$) and let U be the path from (u, j) to (v, i); by (c) of Lemma 8 we have $w^1(U) \geq 0$; hence (u, j) would be fewer edges away from s than (v, i) and its distance would not be larger either, contradicting the choice of (v, i). Hence, on the path from s to (v, i) are no vertices of type $(w, 0), (w, 3)$ and hence no negative edges, which ensures that the distance from s to (v, i) is ≥ 0. □

Now in the Dijkstra algorithm on D'_h, we label all vertices $(v, 0)$ and $(v, 3)$ with 0 and make these labels final. For each arc $((v, 0/3), (u, 1/4))$ we update the distance label of $(u, 1/4)$ if the arc permits to reduce its distance. It is pivotal to notice, that the arc $((v, 0/3), (u, 1/4))$ can not bring another improvement afterward, since its source $(v, 0/3)$ got already a final distance label. After this update, the arcs $((v, 0/3), (u, 1/4))$ can be removed from D'_h. The resulting network has no longer any negative arcs, hence Dijkstra's algorithm can finish the computation of the distance labels. So we have proved:

Lemma 14. *Given $\bar{x} \in CSTAB(G)$ and $h \in G$, a potential for D_h can be computed in time $O(m + n \log n)$.*

This yields the promised result:

Theorem 15. *The separation problem given $\bar{x} \in CSTAB(G)$ for $W_{\mathcal{A}}STAB(G)$ and $W_{\mathcal{B}}STAB(G)$ can be solved in time by $O(n^2)$ applications of the fast Dijkstra algorithm (each requiring $O(m + n \log n)$); of these, $O(n)$ Dijkstra invocations are used to precompute potentials and then $O(n^2)$ invocations are used to search violated wheels.*

5. Conclusion

The categorical product of the original graph with a small gadget presented a new way to keep track of the different parity requirements when separating general wheel inequalities. The advantage of our approach over the approach by Cheng and Cunningham [8, 9], is that our approach preserves the sparsity of the graph, while their approach involves an auxillary graph that is dense. So when facing problems on sparse graphs, the approach via categorical graph product might beneficial.

References

[1] M. W. Padberg, On the facial structure of set packing polyhedra, Mathematical Programming 5 (1973) 199–215.

[2] R. E. Bixby, E. K. Lee, Solving a Truck Dispatching Scheduling Problem using Branch-and-Cut, Operations Research 46 (3) (1998) 355–367.

[3] F. Barahona, A. Weintraub, R. Epstein, Habitat dispersion in forest planning and the stable set problem, Operations Research 40 (1992) S14–S21.

[4] G. L. Nemhauser, G. Sigismondi, A strong cutting plane/branch-and-bound algorithm for node packing, J. Oper. Res. Soc 43 (5) (1992) 443–457.

[5] M. Grötschel, L. Lovász, A. Schrijver, Geometric Algorithms and Combinatorial Optimization, vol. 2 of *Algorithms and Combinatorics*, Springer Verlag, Berlin Heidelberg, Germany, 1. edn., 1988.

[6] M. Grötschel, W. R. Pulleyblank, Weakly bipartite graphs and the max-cut problem, Operations Research Letters 1 (1) (1981) 23–27.

[7] M. L. Fredman, R. E. Tarjan, Fibonacci heaps and their uses in improved network optimization algorithms, Journal of the ACM 34 (3) (1987) 596–615.

[8] E. Cheng, W. H. Cunningham, Separation Problems for the Stable Set Polytope, in: E. Balas, J. Clausen (Eds.), The 4th Integer Programming and Combinatorial Optimization Conference Proceedings, vol. 920 of *Lecture Notes in Computer Science*, Springer, Berlin, 65–79, 1995.

[9] E. Cheng, W. H. Cunningham, Wheel inequalities for stable set polytopes, Mathematical Programming 77 (3) (1997) 389–421.

[10] D. B. Johnson, Efficient Algorithms for Shortest Paths in Sparse Networks, Journal of the ACM 24 (1) (1977) 1–13.