Spaceability of subsets of the disc algebra

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Abstract

In this paper we analyse the topological and linear structure of different subsets of the disc algebra. Among others, we consider the set of functions in the disc algebra having a Taylor series about 0 which is unboundedly divergent on a given subset of the unit circle of vanishing arc length measure, and the subsets of functions having uniformly bounded or uniformly convergent Taylor series on the unit circle.

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1 Introduction

The pathological behaviour of Fourier series of continuous functions has been broadly studied in the past years. Its origin goes back to du Bois-Reymond (1873), who was the first one to exhibit an example of a continuous function on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ whose Fourier series diverges at a point. This was improved by Kahane and Katznelson (1966, see e.g. [17]) by extending the divergence to arbitrary sets $E \subset \mathbb{T}$ of (arc length) measure zero, and thus complementing the famous Carleson Theorem on almost everywhere convergence of Fourier series of functions in L^2 ([11]).

In recent years, these divergence properties have been proved to be topologically generic, that is, not only functions f fulfilling the properties exist, but they hold true for a residual set of functions in the corresponding spaces (see e.g. [16]). Moreover, even linear and algebraic structures can be detected within the set of such functions (see e.g. [2], [4]). We refer to the survey [8] and the book [1] for a wide background on this topic.

Our aim is to find linear structures in various subsets of the disc algebra. We recall some standard notation: Let $\mathbb{D}$ denote the open disc in the complex plane $\mathbb{C}$, so that $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$. For K a compact space, let $C(K)$ be the Banach space of continuous functions $f : K \longrightarrow \mathbb{C}$ endowed with the supremum norm $\|f\|_\infty = \sup_{z\in K} |f(z)|$. We will write $H(\mathbb{D})$ for the family of holomorphic functions on

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the unit disc $\mathbb{D}$. Then, each $f \in H(\mathbb{D})$ can be represented as $f(z) = \sum_{k=0}^{\infty} a_k z^k$ for $z \in \mathbb{D}$, with a_k being the k-th Taylor coefficient with respect to 0. The disc algebra $A(\mathbb{D})$ is defined as the Banach space of functions $f \in H(\mathbb{D})$ which are continuously extendable up to the boundary $\mathbb{T}$. One can identify $A(\mathbb{D})$ with the space $A(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}) \ : f \text{ holomorphic in } \mathbb{D}\}$ with the uniform norm

$$\|f\|_\infty = \max_{z \in \overline{\mathbb{D}}} |f(z)| = \max_{z \in \mathbb{T}} |f(z)| = \sup_{z \in \mathbb{D}} \left| \sum_{k=0}^{\infty} a_k z^k \right|.$$

Fejér's Theorem implies that the polynomials form a dense subset of $A(\mathbb{D})$.

In the first part we will report on the existence of large topological and algebraic structures of subsets of functions in the disc algebra whose Taylor series about 0 are unboundedly divergent on certain subsets of the unit circle. While having mainly survey character, two results complementing the theory are proved. In the second part we search for linear and algebraic structures in subsets of the space of functions in the disc algebra having uniformly bounded or uniformly convergent Taylor series. In the third part we consider subspaces of the disc algebra consisting of functions whose Taylor series about 0 converges on subsets of the unit circle.

We recall some definitions and results that will be used occasionally. If X is a Baire space, a subset A is said to be residual if it contains a dense G_δ-subset of X. Given a vector space X and a subset A, we say that A is maximal lineable whenever there is a vector space M such that $M \setminus \{0\} \subset A$ and the dimension of M equals the dimension of X. Moreover, if X is a topological vector space, then A is said to be spaceable if there is an infinite dimensional closed vector space M with $M \setminus \{0\} \subset A$, and dense-lineable if there is a dense infinite dimensional vector space M with $M \setminus \{0\} \subset A$. If X is a metrizable separable topological vector space, and $Y \subset X$ is a vector subspace, then $X \setminus Y$ is dense-lineable if X is lineable (see [10]). Thus, if $X \setminus Y$ is spaceable, then $X \setminus Y$ is also dense-lineable (and maximal lineable). Finally, when X is a topological vector space contained in some (linear) algebra, then A is called algebrable if there is an algebra M so that $M \setminus \{0\} \subset A$, and M is infinitely generated, that is, the cardinality of any system of generators of M is infinite, and dense-algebrable if the algebra M can be chosen to be dense.

Remark 1.1. Let X and Z be Fréchet spaces, and $T : Z \to X$ be a continuous linear operator with range $Y = T(Z)$ non-closed. Then, according to a result of Kitson and Timoney ([18]), the complement $X \setminus Y$ is spaceable. By considering the inclusion map $T : Y \to X$, for a Fréchet space $Y \neq X$ with Y densely and continuously embedded in X, it is seen that the complement $X \setminus Y$ is spaceable.

Remark 1.2. By assuming the continuum hypothesis, we have that for X a non-separable F-space, and Y a closed separable subspace of X, the set $X \setminus Y$ is maximal lineable.

2 Unbounded divergence

Given a sequence $(w_n)_{n\in\mathbb{N}}$ of complex numbers, we say that $(w_n)_{n\in\mathbb{N}}$ is unboundedly divergent if $\sup_{n\in\mathbb{N}} |w_n| = \infty$. A strengthened version of the Kahane-Katznelson Theorem says that, for each F_σ-set E of vanishing (arc length) measure, a residual set of functions in $C(\mathbb{T})$ exists so that the Fourier series of f is unboundedly divergent in each point of E (see e.g. [16]). Moreover, the set of functions in $C(\mathbb{T})$ having unboundedly divergent Fourier series on E is known to be spaceable and dense-algebrable (see [1, Theorem 6.3.1], [2]). Our aim is to complete the picture by showing that similar results hold for the disc algebra. Thanks to having now a well-elaborated general theory at disposal, it turns out that this is essentially a consequence of the fact that a lemma of Kahane and Katznelson (see [17, Chapter II, Lemma 3.4]), which is the basis of the proof of the Kahane-Katznelson Theorem, holds for algebraic polynomials as well.

We write $S_n(f, \cdot)$ for the n-th partial sum of the Taylor series $\sum_{k=0}^{\infty} a_k z^k$ of f, that is, $S_n(f, z) = \sum_{k=0}^{n} a_k z^k$.

Theorem 2.1. *Let $E \subset \mathbb{T}$ be a F_σ-set of measure zero. Then, for a residual set of functions $f \in A(\mathbb{D})$, the sequence $(S_n(f, z))_n$ is unboundedly divergent for each $z \in E$.*

Theorem 2.2. *Let $E \subset \mathbb{T}$ be of measure zero. Then, the family $\mathcal{A}$ of all functions $f \in A(\mathbb{D})$ such that the partial sums series $(S_n(f, z))_n$ is unboundedly divergent for every $z \in E$ is spaceable and dense-algebrable in $A(\mathbb{D})$.*

The proofs of the above theorems are based on two general results of Bayart. The first one ([4, Theorem 2]) focuses on topological genericity in Banach spaces:

Theorem 2.3 (Bayart, 2005)**.** *Let X be a Banach space and E a σ-compact topological space. Suppose that for each $t \in E$ and each $n \geq 0$ a linear form $\varphi_n(\cdot, t)$ is given such that $\varphi_n : X \times E \longrightarrow \mathbb{C}$ is continuous. For $g \in X$ and $t \in E$ let*

$$\delta_N(g, t) = \sup_{n>N} |\varphi_n(g, t) - \varphi_N(g, t)|.$$

Suppose that, for all $M, N > 0$ and each compact subset K of E, there exists $g \in X$ such that $\|g\|_X \leq 1$ and $\delta_N(g, t) > M$ for every $t \in K$. Then, for a residual set of functions $f \in X$, the sequence $(\varphi_n(f, t))_{n\geq 0}$ is unboundedly divergent for each $t \in E$.

Theorem 2.1 is mainly a consequence of Theorem 2.3 combined with the following auxiliary result, which is a variation of the lemma of Kahane and Katznelson mentioned above. The result follows from the corresponding lemma by replacing the anti-holomorphic polynomial $\varphi = \varphi_F$ constructed in their proof (see [17, Chapter II, Lemma 3.4]) by the holomorphic polynomial $Q_F := \overline{\varphi}_F$.

Lemma 2.4. *Let $F \subset \mathbb{T}$ be a union of a finite number of subarcs of $\mathbb{T}$, and denote the measure of F by δ. There exists a polynomial Q_F such that*

(i) $\sup_{n\in\mathbb{N}} |S_n(Q_F, z)| > \dfrac{1}{2\pi} \log\left(\dfrac{1}{3\delta}\right)$ *on F.*

(ii) $\|Q_F\|_\infty \le 1$.

As an immediate consequence we have

Lemma 2.5. *Let $E \subset \mathbb{T}$ be of measure zero. Then, for all $M > 0$, there exists a polynomial Q with $\|Q\|_\infty \le 1$ and $\sup_{n\in\mathbb{N}} |S_n(Q, z)| > M$ for all $z \in E$.*

Proof. Let $M > 0$ and let $\delta > 0$ be so that $\log((3\delta)^{-1}) > 2\pi M$. Since E has measure zero, we can find a finite union F of subarcs of $\mathbb{T}$ with $E \subset F$, and having measure less than δ. If $Q = Q_F$ is as in Lemma 2.4, then $\|Q\|_\infty \le 1$ and $\sup_{n\in\mathbb{N}} |S_n(Q, z)| > M$ for all $z \in E$. □

Proof of Theorem 2.1. Let $X = A(\mathbb{D})$. For all $M, N > 0$ define

$$g(z) := z^{N+1}Q(z) \quad \text{on } \mathbb{D},$$

where Q is the polynomial provided by Lemma 2.5, that is, $\|Q\|_\infty \le 1$ and

$$\sup_{n\in\mathbb{N}} |S_n(Q, z)| > M \quad \text{on } E.$$

We have then $\|g\|_\infty \le 1$, and

$$\sup_{n>N} |S_n(g, z) - S_N(g, z)| = \sup_{n>N} |S_n(g, z)| = \sup_{m\in\mathbb{N}_0} |S_m(Q, z)| > M \quad \text{on } E.$$

Now, we just apply Theorem 2.3 with $X = A(\mathbb{D})$ and $\varphi_n(g, \cdot) = S_n(g, \cdot)$, and the proof is complete. □

Recall that a sequence $(u_n)_n$ in a Banach space X is called a basic sequence if, for each u belonging to $M = \overline{\text{span}}\{u_n : n \in \mathbb{N}\}$, there exists a sequence $(\alpha_n)_n$ of scalars such that $u = \sum_{n=1}^{\infty} \alpha_n u_n$. The coefficient functionals u_k^* are defined by

$$u_k^*\left(\sum_{n=1}^{\infty} \alpha_n u_n\right) = \alpha_k, \quad k \in \mathbb{N}.$$

They are continuous on M, and can be extended to X by the Hahn-Banach theorem.

Following a construction provided by Bayart in [4, Theorem 3] and [3, Theorem 3], we obtain for an arbitrary Banach space $X \subset L^1(\mathbb{T})$:

Theorem 2.6. *Let $X \subset L^1(\mathbb{T})$ be a Banach space so that the polynomials form a dense subset of X, and let $E \subset \mathbb{T}$. Suppose that $\|z^n f\|_X = \|f\|_X$ for all $n \in \mathbb{N}$ and $f \in X$, and that for all $M > 0$ there exists a polynomial Q such that $\|Q\|_X \le 1$ and $\sup_{n\in\mathbb{N}} |S_n(Q, z)| > M$ for each $z \in E$. Moreover, let $\mathcal{A}$ be the family of all functions $f \in X$ such that the sequence $(S_n(f, z))_n$ is unboundedly divergent for every $z \in E$. Then, $\mathcal{A}$ is dense-lineable in X, and if, in addition, there exists a basic sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that $\|u_n^*\| \le 1$ for all $n \in \mathbb{N}$, then $\mathcal{A}$ is spaceable in X.*

Proof of Theorem 2.2. According to Lemma 2.5, for $X = A(\mathbb{D})$, given any $M > 0$, there exists a polynomial Q with $\|Q\|_\infty \leq 1$ and $\sup_{n\in\mathbb{N}} |S_n(Q,z)| > M$ for all $z \in E$. Moreover, the sequence $u_n(z) = z^{2^n}$ is a basic sequence for $A(\mathbb{D})$ (see e.g. [17, Chapter V, Theorem 1.4]). Now, Theorem 2.6 implies the spaceability of $\mathcal{A}$ in $A(\mathbb{D})$. Finally, using Lemma 2.5 instead of the corresponding (Kahne-Katznelson) lemma for trigonometric polynomials in Section 2.1 of [2], essentially the same proof as of [2, Theorem 2.1] leads to the dense-algebrability of our family $\mathcal{A}$. □

Various recent work focusses on strengthened notions of unbounded divergence. In [5] (see also [6]) it is shown that for a residual set of functions in $C(\mathbb{T})$, the partial sums of the Fourier series diverge on sets of Hausdorff dimension 1 with a maximal rate of growth.

A different kind of maximal divergence can be formulated in terms of universality. A formal series $\sum_{k=0}^{\infty} f_k$ of functions $f_k \in C(\mathbb{T})$ is said to be pointwise universal on a set $E \subset \mathbb{T}$ if, for all Baire class 1 functions $h : E \to \mathbb{C}$, a subsequence of the partial sums converges to h pointwise on E. The series is called uniformly universal on E if E is closed and the partial sums form a dense set in $C(E)$. In [20] and [7], respectively, for arbitrary countable subsets E of $\mathbb{T}$, residuality and maximal dense-lineabilty, as well as spaceability of the set of functions in $C(\mathbb{T})$ having pointwise universal Fourier series on E were proved. In [15] and [9], similar results for the disc algebra and Taylor series were added. As a consequence of the latter, in [9] also residuality, maximal dense-lineability and spaceability of the set of functions in $A(\mathbb{D})$ having uniformly universal Taylor series on a residual set in the hyperspace of compact sets in $\mathbb{T}$ are proved. Generically, sets in the hyperspace of $\mathbb{T}$ are perfect sets, and, as such, in particular locally uncountable. On the other hand, it is known that sets of uniform universality are severely restricted in that they have to be strongly porous at all points ([9]; cf. also [21]).

Combining universality and convergence, in [21] it is shown that for arbitrary countable $E \subset \mathbb{T}$ functions in the disc algebra exist with Taylor series being pointwise universal on E and converging pointwise except for a set of vanishing Hausdorff dimension. For finite E, the convergence can take place locally uniformly on $\mathbb{T} \setminus E$.

3 Uniformly bounded series

In this section we consider the space $A_{ub}(\mathbb{D})$ of all functions f in the disc algebra whose partial sums $(S_m(f,\cdot))_m$ are uniformly bounded with respect to the uniform norm, that is

$$A_{ub}(\mathbb{D}) = \{f \in A(\mathbb{D}) \ : \ (\|S_m(f,\cdot)\|_\infty)_m \text{ is bounded}\}.$$

We endow this space with the natural metric

$$\|f\|_{ub} := \sup_{m\in\mathbb{N}_0} \|S_m(f,\cdot)\|_\infty.$$

Since $\|f\|_\infty \le \|f\|_{ub}$ for $f \in A_{ub}(\mathbb{D})$, the space $A_{ub}(\mathbb{D})$ is continuously embedded in $A(\mathbb{D})$.

Lemma 3.1. *$(A_{ub}(\mathbb{D}), \|\cdot\|_{ub})$ is a Banach space.*

Proof. Let $(f_n)_n \subset A_{ub}(\mathbb{D})$ be a Cauchy sequence. Then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $r, s \ge N$ and $m \in \mathbb{N}_0$ we have

$$\|S_m(f_r - f_s, \cdot)\|_\infty < \varepsilon.$$

Since $(f_n)_n$ is a Cauchy sequence in $A(\mathbb{D})$, there exists $f \in A(\mathbb{D})$ such that $\|f - f_n\|_\infty \to 0$ as $n \to +\infty$. From Cauchy's estimates, we have $|a_k| \le \|f\|_\infty$ for $k \ge 0$, and then

$$|S_m(f,z)| \le \sum_{k=0}^{m} |a_k| \le (m+1)\|f\|_\infty.$$

Hence, for fixed $m \in \mathbb{N}_0$, the mapping $T_m f = S_m(f, \cdot)$ is continuous from $A(\mathbb{D})$ to $A(\mathbb{D})$. Now, for $n \ge N$ and $m \in \mathbb{N}_0$

$$\varepsilon > \|S_m(f_r - f_N, \cdot)\|_\infty \to \|S_m(f - f_N, \cdot)\|_\infty$$

as $r \to +\infty$, hence $\|S_m(f - f_n, \cdot)\|_\infty \le \varepsilon$ for $n \ge N$ and $m \in \mathbb{N}_0$. This means that

$$\|f - f_n\|_{ub} \le \varepsilon$$

for $n \ge N$. Finally with $M := \sup_m \|S_m(f_N, \cdot)\|_\infty$ we obtain that

$$\|S_m(f,\cdot)\|_\infty \le \|S_m(f - f_N, \cdot)\|_\infty + \|S_m(f_N, \cdot)\|_\infty < \varepsilon + M$$

for all $m \in \mathbb{N}_0$, which shows that $f \in A_{ub}(\mathbb{D})$. □

Since, by definition, for $f \in A_{ub}(\mathbb{D})$ we have uniform boundedness of the sequence partial sums $S_n f$ on $\overline{\mathbb{D}}$, the set $\mathcal{A}$ from Theorem 2.2 is contained in $A(\mathbb{D}) \setminus A_{ub}(\mathbb{D})$. Thus, Theorem 2.2 implies

Theorem 3.2. *The set $A(\mathbb{D}) \backslash A_{ub}(\mathbb{D})$ is spaceable and dense-algebrable in $A(\mathbb{D})$.*

As a consequence of the following result of Erdös, Herzog and Piranian ([12]), we will show that the space $A_{ub}(\mathbb{D})$ is not separable.

Theorem 3.3 (Erdös, Herzog and Piranian, 1954)**.** *Let $E \subset \mathbb{T}$ be a F_σ-set of logarithmic measure zero. Then, there exists a function $f \in A_{ub}(\mathbb{D})$ so that $S_n(f, \cdot)$ diverges on E and converges on $\mathbb{T} \backslash E$.*

Theorem 3.4. *The space $(A_{ub}(\mathbb{D}), \|\cdot\|_{ub})$ is not separable.*

Proof. According to Theorem 3.3, there exists a function $f \in A_{ub}(\mathbb{D})$ such that $(S_n(f,1))_n$ diverges and $(S_n(f,z))_n$ converges for all $z \in \mathbb{T} \backslash \{1\}$. We choose $M > 0$ such that for all $N \in \mathbb{N}$ there are $n, m \ge N$ with

$$|S_n(f,1) - S_m(f,1)| \ge M. \tag{1}$$

Now, given $\zeta \in \mathbb{T}$, we can consider the functions $f_\zeta \in A_{ub}(\mathbb{D})$ given by

$$f_\zeta(z) := f\left(\overline{\zeta} z\right),$$

that is, f_ζ is the rotation of f via $\zeta \in \mathbb{T}$, and so f_ζ has the same behaviour at the point $\zeta \in \mathbb{T}$ as f at the point 1. We consider the family $\mathcal{F} := \{f_\zeta : \zeta \in \mathbb{T}\}$ in $A_{ub}(\mathbb{D})$, which is uncountable, and such that for each ζ the sequence $(S_n(f_\zeta, \cdot))_n$ converges on $\mathbb{T} \setminus \{\zeta\}$ and satisfies (1) at the point ζ.

By way of contradiction, assume that $A_{ub}(\mathbb{D})$ is separable, that is, there exists $D \subset A_{ub}(\mathbb{D})$ countable and dense. We write $D := \{g_k : k \in \mathbb{N}\}$, and put $\varepsilon := M/6$. Then, for all $\zeta \in \mathbb{T}$, there exists $k(\zeta) \in \mathbb{N}$ such that

$$\|f_\zeta - g_{k(\zeta)}\|_{ub} = \sup_{m \in \mathbb{N}_0} \|S_m(f_\zeta - g_{k(\zeta)}, \cdot)\|_\infty < \varepsilon.$$

In particular, for all $m \in \mathbb{N}_0$ and $z \in \mathbb{T}$ we have

$$|S_m(f_\zeta, z) - S_m(g_{k(\zeta)}, z)| < \varepsilon.$$

Since the family $\mathcal{F}$ is uncountable, and D is countable, there must exist $\zeta, \zeta' \in \mathbb{T}$ with $\zeta' \neq \zeta$, and such that $k(\zeta) = k(\zeta') =: k$. Then, for $z = \zeta$ and n, m satisfying (1) we obtain

$$\begin{aligned} |S_n(g_k, \zeta) - S_m(g_k, \zeta)| \quad &\geq \quad |S_n(f_\zeta, \zeta) - S_m(f_\zeta, \zeta)| - |S_m(f_\zeta - g_k, \zeta)| \\ &\quad - |S_n(f_\zeta - g_k, \zeta)| \geq M - 2\varepsilon = 2M/3. \end{aligned}$$

Thus, the function $g_k \in A_{ub}(\mathbb{D})$ has the same property at the point $\zeta \in \mathbb{T}$ as the function f_ζ, that is, the sequence $(S_n(g_k, \zeta))_n$ is divergent.

On the other hand, for $n, m \in \mathbb{N}$ so large that $|S_m(f_{\zeta'}, \zeta) - S_m(f_{\zeta'}, \zeta)| < \varepsilon$, we have that

$$\begin{aligned} |S_n(g_k, \zeta) - S_m(g_k, \zeta)| \quad &\leq \quad |S_n(g_k, \zeta) - S_n(f_{\zeta'}, \zeta)| + |S_n(f_{\zeta'}, \zeta) - S_m(f_{\zeta'}, \zeta)| \\ &\quad + |S_m(f_{\zeta'}, \zeta) - S_m(g_k, \zeta)| < 3\varepsilon = M/2. \end{aligned}$$

Hence, the partial sums of g_k would have at the same time a variation greater than $2M/3$ and smaller than $M/2$ at the point $\zeta \in \mathbb{T}$, which is clearly a contradiction. Thus, the space $A_{ub}(\mathbb{D})$ is not separable. □

We denote by $A_{uc}(\mathbb{D})$ the space of all functions in the disc algebra having uniformly convergent Taylor series on $\mathbb{T}$, that is

$$A_{uc}(\mathbb{D}) := \{f \in A(\mathbb{D}) : \sum_{k=0}^{\infty} a_k z^k \text{ converges uniformly on } \mathbb{T}\}.$$

Note that, according to the maximum principle, for all $f \in A_{uc}(\mathbb{D})$ the sequence of partial sums $S_n(f, \cdot)$ is a uniform Cauchy sequence on $\overline{\mathbb{D}}$, and therefore converges to f uniformly on $\overline{\mathbb{D}}$ as $n \to \infty$. With the norm $\|\cdot\|_{ub}$ the space $A_{uc}(\mathbb{D})$ is completely metrized:

Lemma 3.5. *$A_{uc}(\mathbb{D})$ is closed in $A_{ub}(\mathbb{D})$.*

Proof. Let $(f_n)_n$ be a sequence in $A_{uc}(\mathbb{D})$ with $\|f_n - f\|_{ub} \to 0$. Then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and all $m \in \mathbb{N}_0$, we have $\|f_n - f\|_\infty < \varepsilon$, and $\|S_m(f - f_n, \cdot)\|_\infty < \varepsilon$. This implies

$$\begin{aligned} \|f - S_m(f,\cdot)\|_\infty &\leq \|f - f_N\|_\infty + \|f_N - S_m(f_N,\cdot)\|_\infty + \|S_m(f_N - f,\cdot)\|_\infty \\ &< 2\varepsilon + \|f_N - S_m(f_N,\cdot)\|_\infty \end{aligned}$$

for all $m \in \mathbb{N}_0$. Since $(S_m(f_N,\cdot))_m$ converges uniformly to f_N, we obtain that $\|f - S_m(f,\cdot)\|_\infty < 3\varepsilon$ for m sufficiently large. □

Lemma 3.6. *For all $f \in A_{uc}(\mathbb{D})$ the partial sums $S_n(f,\cdot)$ converge to f in $A_{uc}(\mathbb{D})$. In particular, $A_{uc}(\mathbb{D})$ is the closure of the polynomials in $A_{ub}(\mathbb{D})$.*

Proof. Since $(S_n(f,\cdot))$ tends to f uniformly on $\overline{\mathbb{D}}$, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $m > n \geq N$ we have

$$\|S_m(f,\cdot) - S_n(f,\cdot)\|_\infty < \varepsilon.$$

The projection property of the S_n implies that

$$\sup_{m\in\mathbb{N}_0} \|S_m(f - S_n(f,\cdot),\cdot)\|_\infty = \sup_{m\in\mathbb{N}_0,\, m>n} \|S_m(f,\cdot) - S_n(f,\cdot)\|_\infty \leq \varepsilon$$

for all $n \geq N$. □

Since the space $A_{uc}(\mathbb{D})$ is continuously embedded in $A(\mathbb{D})$, and recalling that $A_{uc}(\mathbb{D})$ is a closed separable subspace of $A_{ub}(\mathbb{D})$, by Remark 1.2 we obtain:

Corollary 3.7. *The complement $A_{ub}(\mathbb{D})\backslash A_{uc}(\mathbb{D})$ is maximal lineable.*

Let now

$$A_{abs}(\mathbb{D}) := \{f \in A(\mathbb{D}) : \sum_{k=0}^{\infty} |a_k| < +\infty\}$$

be the analytic Wiener algebra of all functions in the disc algebra having absolutely convergent Taylor series (also denoted by W^+), endowed with the norm

$$\|f\|_{abs} := \sum_{k=0}^{\infty} |a_k| \,.$$

According to the Denjoy-Lusin theorem (see e.g [23, p. 232]), a function f belongs to $A_{abs}(\mathbb{D})$ if $(S_n(f,\cdot))_n$ converges absolutely on a set of positive measure. Denoting by ℓ_1 the space of absolutely summable sequences, via the mapping $\ell_1 \ni (a_n)_{n\in\mathbb{N}_0} \mapsto \sum_{k=0}^{\infty} a_k z^k \in A_{abs}(\mathbb{D})$ the space $A_{abs}(\mathbb{D})$ is isometrically isomorphic to ℓ_1, and hence a Banach space. Since $\|f\|_{ub} \leq \|f\|_{abs}$ for $f \in A_{abs}(\mathbb{D})$, the space $A_{abs}(\mathbb{D})$ is continuously embedded in $A_{uc}(\mathbb{D})$.

As a consequence of the result from Remark 1.1, Kitson and Timoney obtained that the set $A(\mathbb{D})\backslash A_{abs}(\mathbb{D})$ is spaceable. This holds even for the difference between $A_{uc}(\mathbb{D})$ and $A_{abs}(\mathbb{D})$:

Theorem 3.8. *The complement $A_{uc}(\mathbb{D})\backslash A_{abs}(\mathbb{D})$ is spaceable in $A_{uc}(\mathbb{D})$.*

Proof. As noted above, $A_{abs}(\mathbb{D})$ is continuously embedded in $A_{uc}(\mathbb{D})$. By Fejér's or Hardy's example (see [19]), $A_{uc}(\mathbb{D}) \neq A_{abs}(\mathbb{D})$. Since the polynomials are dense in $A_{uc}(\mathbb{D})$, also $A_{abs}(\mathbb{D})$ is dense in $A_{uc}(\mathbb{D})$. According to Remark 1.1, the proof is complete. □

We close this part with a remark concerning conformal mappings in the disc algebra. If $f \in H(\mathbb{D})$ is injective, then Carathéodory's Theorem shows that $f \in A(\mathbb{D})$ if and only if $f(\mathbb{D})$ is a bounded simply connected domain with locally connected boundary. By Abel's Theorem and the Fejér Tauberian theorem (see e.g. [19, p. 65]), this is also equivalent to $f \in A_{uc}(\mathbb{D})$. Note that f does not need to be injective on $\mathbb{T}$, so this subclass of $A(\mathbb{D})$ is not easily detected by the corresponding boundary functions.

Moreover, there are functions in $A(\mathbb{D})$ which are injective even on $\overline{\mathbb{D}}$, and which do not belong to $A_{abs}(\mathbb{D})$ (see [22]), so $A_{uc}(\mathbb{D})\backslash A_{abs}(\mathbb{D})$ contains functions which map $\mathbb{D}$ conformally onto a Jordan domain.

4 Uniform convergence on subsets

So far we have detected linear structures in the partition

$$A_{abs}(\mathbb{D}) \subset A_{uc}(\mathbb{D}) \subset A_{ub}(\mathbb{D}) \subset A(\mathbb{D}).$$

We are now interested in studying what happens if we reduce the set of uniform convergence. If $f \in A(\mathbb{D})$ has modulus of continuity $o(\log \delta^{-1})$ on some closed arc $B \subset \mathbb{T}$, then the local Dini-Lipschitz theorem (see e.g. [23, p. 63]) shows that $(S_n(f,\cdot))_n$ converges to f locally uniformly on the corresponding open arc.

We start with functions in the disc algebra that have only one singularity on $\mathbb{T}$, which we suppose to be the point 1. Then f is smooth on $\mathbb{T}$, and, in particular, $(S_n(f,\cdot))_n$ converges to f locally uniformly on $\mathbb{T} \setminus \{1\}$.

If $C_\delta = \{z : |\arg(z-1)| \leq \delta\}$ is the sector with vertex at 1 and angle $2\delta < \pi$, a result of M. Riesz (see e.g. [19, p. 64]) shows that $f \in A_{uc}(\mathbb{D})$ if $f \in A(\mathbb{D})$ extends holomorphically to $r\mathbb{D} \setminus C_\delta$, for some $r > 1$. The situation changes drastically if the cone is replaced by a circle touching $\mathbb{T}$ at the point 1, as the following result from [13] shows. Here, $B_\delta(\zeta) = \{z \in \mathbb{C} \,:\, |z-\zeta| \leq \delta\}$ for $\delta > 0$ and $\zeta \in \mathbb{C}$.

Theorem 4.1 (Gaier, 1952)**.** *For arbitrary $\rho > 1$ there exist a function $f \in A(\mathbb{D})$ which extends holomorphically to $\mathbb{C} \setminus B_{\rho-1}(\rho)$ and having the property that $(S_n(f,1))_n$ is divergent.*

Let X, Y be Banach spaces, and let $(T_n)_n$ be a sequence of continuous linear operators $T_n : X \to Y$. We recall the Banach-Steinhaus Theorem saying that pointwise convergence of $(T_n)_n$ on X is equivalent to boundedness of the norms $\|T_n\|$ and pointwise convergence of $(T_n)_n$ on some dense set in X.

We consider now the compact sets

$$K_r := r\overline{\mathbb{D}} \setminus \{z : |z - r| < r - 1\},$$

where $r > 1$, and the continuous linear functionals $T_{n,r} : A(K_r) \to \mathbb{C}$ with $T_{n,r} f := S_n(f, 1)$.

Proposition 4.2. *The sequence* $(\|T_{n,r}\|)_n$ *is unbounded.*

Proof. Assume that the sequence $(\|T_{n,r}\|)_n$ is bounded. Since K has connected complement, according to Mergelian's Theorem, the set of polynomials is dense in $A(K_r)$. Moreover, $(T_{n,r}P)_n$ converges for all polynomials P (because $S_n(P, 1) = P(1)$ for n large enough). But then the sequence $(T_{n,r}f)_n$ converges for all $f \in A(K_r)$, which is not the case by Gaier's example (Theorem 4.1), for $\rho < r$. □

The following lemma is a variant of the uniform boundedness principle in the case of functionals.

Lemma 4.3. *Let X be a Banach space and let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in the dual of X. If $(\|\varphi_n\|)$ is unbounded and (φ_n) converges pointwise on a dense set in X, then for a residual set of $x \in X$ the sequence $(\varphi_n x)$ forms a dense set in the scalar field $\mathbb{K}$.*

Proof. According to the Universality Criterion (see [14]), it suffices to show that, for all $z \in X$, $c \in \mathbb{K}$ and $\varepsilon > 0$, there are $x \in X$ and $m \in \mathbb{N}$ such that $\|x-z\| < \varepsilon$ and $|\varphi_m x - c| < \varepsilon$.

By assumption, $y \in X$, $d \in \mathbb{K}$ and N exist with $\|y-z\| < \varepsilon/2$ and $|\varphi_n y - d| < \varepsilon$ for all $n \geq N$. Moreover, the uniform boundedness principle assures that the set of $u \in X$ with $\sup_n |\varphi_n u| = \infty$ is residual in X. Therefore, there is $u \in X$ with $\|u\| < \varepsilon/2$ and $|\varphi_m u| > |c - d|$ for some $m \in \mathbb{N}$ with $m \geq N$. Then

$$v := \frac{c - d}{\varphi_m u} \cdot u$$

satisfies $\varphi_m v = c - d$ and $\|v\| \leq \|u\| < \varepsilon/2$, and thus for $x := y + v$ we obtain $\|x - z\| < \varepsilon$ and $|\varphi_m x - c| = |\varphi_m y - d| < \varepsilon$. □

Proposition 4.4. *Let $r > 1$. For a residual set of functions $f \in A(K_r)$ the partial sums $S_n(f, 1)$ are dense in $\mathbb{C}$.*

Proof. We consider $\varphi_n = T_{n,r}$ as above. Since $(\|\varphi_n\|)$ is unbounded and since $(\varphi_n P)$ is eventually constant for all polynomials P, Mergelian's Theorem and Lemma 4.3 imply the assertion. □

Consider now a non-empty proper closed subset B of the unit circle and the space

$$A_{uc}(B) := \{f \in A(\mathbb{D}) \;:\; \sum_{k=0}^{\infty} a_k z^k \text{ converges uniformly on } B\},$$

endowed with the norm

$$\|f\|_{uc,B} := \|f\|_\infty + \sup_{m\in\mathbb{N}_0} \|S_m(f,\cdot)\|_{\infty,B},$$

where $\|f\|_{\infty,B} := \sup_{z\in B} |f(z)|$. With essentially the same proof as for Lemma 3.1 and 3.5 it is seen that $(A_{uc}(B), \|\cdot\|_{uc,B})$ is a Banach space.

For $f \in A(\mathbb{D})$ and $r \in (0,1)$ we define the functions

$$f_r(z) := f(rz).$$

Clearly, $f_r \in A(r^{-1}\overline{\mathbb{D}}) \subset A_{abs}(\mathbb{D})$ for $0 < r < 1$.

Lemma 4.5. *Let $f \in A_{uc}(B)$. Then, the functions f_r for $r \in (0,1)$ converge in $\|\cdot\|_{uc,B}$-norm to f when $r \to 1^-$. Moreover, the polynomials are dense in $A_{uc}(B)$.*

Proof. 1. We show the convergence of f_r to f as $r \to 1^-$ in the $\|\cdot\|_{uc,B}$ norm. Firstly, we have

$$\begin{aligned} S_m(f(z) - f_r(z)) &= \sum_{k=0}^{m} a_k z^k - \sum_{k=0}^{m} a_k r^k z^k = \sum_{k=0}^{m} a_k z^k (1 - r^k) \\ &= (1-r) \sum_{k=0}^{m} a_k z^k \sum_{\nu=0}^{k-1} r^\nu = (1-r) \sum_{\nu=0}^{m-1} r^\nu \sum_{k=\nu+1}^{m} a_k z^k \\ &= (1-r) \sum_{\nu=0}^{m-1} r^\nu (S_m(f,z) - S_\nu(f,z)). \end{aligned}$$

Since $S_m(f,\cdot) \to f$ (as $m \to \infty$) uniformly on B, the sequence $(S_m(f,\cdot))_m$ is uniformly Cauchy on B, that is, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for all $m \geq \nu \geq N$ we have

$$\|S_m(f,\cdot) - S_\nu(f,\cdot)\|_{\infty,B} < \varepsilon.$$

If we choose $r(\varepsilon) < 1$ such that $2\|f\|_{uc,B}(1 - r(\varepsilon)^N) < \varepsilon$, we obtain that for all r with $r(\varepsilon) < r < 1$, and all $m \in \mathbb{N}_0$

$$\begin{aligned} \|S_m(f - f_r, \cdot)\|_{\infty,B} &\leq 2\|f\|_{uc,B}(1-r) \sum_{\nu=0}^{N-1} r^\nu + (1-r) \sum_{\nu=N}^{m-1} r^\nu \varepsilon \\ &\leq 2\|f\|_{uc,B}(1 - r^N) + \varepsilon(1 - r^m) \leq 2\varepsilon\,. \end{aligned}$$

Furthermore, the uniform continuity of the functions f_r implies that

$$\|f - f_r\|_\infty \to 0$$

as $r \to 1^-$. Thus, $\|f - f_r\|_{uc,B} \to 0$, and the proof is complete.

2. The first part implies that $A_{abs}(\mathbb{D})$ is dense in $A_{uc}(B)$. Since the polynomials are dense in $A_{abs}(\mathbb{D})$, and since $\|f\|_{uc,B} \leq 2\|f\|_{abs}$ for all $f \in A_{abs}(\mathbb{D})$, the polynomials are also dense in $A_{uc,B}(B)$. □

Since the Taylor series from Gaier's example (Theorem 4.1) converges locally uniformly on $\mathbb{T} \setminus \{1\}$, we have $A_{uc}(B) \neq A_{uc}(\mathbb{D})$ for all closed sets $B \subset \mathbb{T} \setminus \{1\}$. By rotation, the same is true for each proper closed subset B of $\mathbb{T}$. As a consequence, with $A_{uc}(\mathbb{T}) := A_{uc}(\mathbb{D})$ and $A_{uc}(\varnothing) := A(\mathbb{D})$ we obtain

Theorem 4.6. *Let $B_1 \subset B_2 \subset \mathbb{T}$ be closed sets with $B_1 \neq B_2$. Then the complement $A_{uc}(B_1) \backslash A_{uc}(B_2)$ is spaceable in $A_{uc}(B_1)$. In particular, $A(\mathbb{D}) \backslash A_{uc}(\{1\})$ is spaceable in $A(\mathbb{D})$.*

Proof. Since $\|f\|_{uc,B} \leq 2\|f\|_{ub}$ for $f \in A_{uc}(\mathbb{T})$, the space $A_{uc}(\mathbb{T})$ is continuously embedded in $A_{uc}(B)$. By definition, $A_{uc}(B_2)$ is continuously embedded in $A_{uc}(B_1)$ if $B_2 \neq \mathbb{T}$. If $\zeta \in B_2 \setminus B_1$, then there exists some $f \in A_{uc}(B_1)$ such that $S_n(f, \zeta)$ is divergent. This implies $A_{uc}(B_1) \neq A_{uc}(B_2)$. Since $A_{abs}(\mathbb{D})$ is dense in $A_{uc}(B_1)$, also $A_{uc}(B_2)$ is dense in $A_{uc}(B_1)$. Now, the result follows from Remark 1.1. □

If U is a proper open subset of $\mathbb{T}$, and $(B_k)_k$ is an increasing sequence of closed sets with $\bigcup_k B_k = U$, then $A_{uc}(U) := \bigcap_k A_{uc}(B_k)$ equipped with the sequence of norms $(|| \cdot ||_{uc,B_k})$ is a Fréchet space. Since, by Gaier's example, $A_{uc}(U) \neq A_{uc}(\overline{U})$, we obtain from Remark 1.1, in a similar way as above,

Theorem 4.7. *The set $A_{uc}(U) \setminus A_{uc}(\overline{U})$ is spaceable in $A_{uc}(U)$. In particular, $A_{uc}(\mathbb{T} \setminus \{1\}) \setminus A_{uc}(\mathbb{D})$ is spaceable in $A_{uc}(\mathbb{T} \setminus \{1\})$.*

According to a result in [21] already mentioned at the end of the first section, for finite $E \subset \mathbb{T}$, functions in $A_{uc}(\mathbb{T} \setminus E)$ with Taylor series universal on E exist.

References

[1] R.M. Aron, L. Bernal-González, D. Pellegrino and J.B. Seoane-Sepúlveda: Lineability: The search for linearity in Mathematics, Monographs and Research Notes in Mathematics. Chapman & Hall / CRC, Boca Raton (2016).

[2] R.M. Aron, D. Pérez-García and J.B. Seoane-Sepúlveda, *Algebrability of the set of non-convergent Fourier series*, Studia Math. **175** (2006), 83-90.

[3] F. Bayart, *Linearity of sets of strange functions*, Michigan Math. J. **53** (2005), no. 2, 291–303.

[4] F. Bayart, *Topological and algebraic genericity of divergence and universality*, Studia Math. **167** (2005), 153-160.

[5] F. Bayart, Y. Heurteaux, *Multifractal analysis of the divergence of Fourier series*. Ann. Sci. c. Norm. Supr. (4) **45** (2012), 927-946 (2013).

[6] F. Bayart, Y. Heurteaux, *Multifractal analysis of the divergence of Fourier series: the extreme cases.* J. Anal. Math. 124 (2014), 387-408

[7] L. Bernal-González, *Lineability of universal divergence of Fourier series*, Integr. Equ. Oper. Theory, **74** (2012) 271-279.

[8] L. Bernal-González, D. Pellegrino and J.B. Seoane-Sepúlveda, *Linear subsets of nonlinear sets in topological vector spaces*, Bull. Amer. Math. Soc. (N.S.) **51** (2014), 71–130.

[9] L. Bernal-González, A. Jung and J. Müller, *Banach spaces of universal Taylor series in the disc algebra*. Integr. Equ. Oper. Theory **86**, 111 (2016).

[10] L. Bernal-González, M. Ordóñez Cabrera, *Lineability criteria, with applications*, J. Funct. Anal. **266** (2014), 39974025.

[11] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966), 135157.

[12] P. Erdös, F. Herzog, G. Piranian, *Sets of divergence of Taylor series and of trigonometric series*, Math. Scand. **2** (1954), 262-266.

[13] D. Gaier, *Über die Summierbarkeit beschr änkter und stetiger Potenzreihen an der Konvergenzgrenze*. Math. Z. **56**, 326334 (1952).

[14] K.G. Grosse-Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. (1999), 345-381.

[15] G. Herzog, P.C. Kunstmann, *Universally divergent Fourier series via Landau's extremal functions* Comment. Math. Carolinae **56** (2015), 159-168.

[16] J.P. Kahane, *Baire's category theorem and trigonometric series*, Journal d'analyse Math. (2000), 143-182.

[17] V. Katznelson: An Introduction to Harmonic Analysis, 3rd Edition, Cambridge University Press (2004).

[18] D. Kitson and R.M. Timoney, *Operator ranges and spaceability*, J. Math. Anal. Appl. **378** (2011), 680-686.

[19] E. Laudau and D. Gaier: Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie. (German) [Presentation and explanation of some more recent results in function theory] 3rd ed. Springer, Berlin (1986).

[20] J. Müller, *Continuous function with universally divergent Fourier series*, C. R. Math. Acad. Sci. Paris, **348** (2010), 1155-1158.

[21] C. Papachristodoulos, M. Papadimitrakis, *On universality and convergence of the Fourier series of functions in the disc algebra*. J. Anal. Math. **137** (2019), 57-71.

[22] G. Piranian, *Jordan domains and absolute convergence of power series*, Michigan Math. J., **9** (1962), 125-128.

[23] A. Zygmund: Trigonometric Series Vol I and II, 3nd ed. Cambridge University Press, Cambridge (2002).

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