Convolution operators on spaces of holomorphic functions

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Abstract

A class of convolution operators on spaces of holomorphic functions related to the Hadamard multiplication theorem for power series and generalizing infinite order Euler differential operators is introduced and investigated. Emphasis is placed on questions concerning injectivity, denseness of range and surjectivity of the operators.

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1 Introduction

Let $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$ denote the punctured plane. Then $\mathbb{C}_*$ is a group with respect to multiplication. Moreover, let U be an open subset of $\mathbb{C}_*$ and let $H(U)$ denote the space of functions holomorphic in U with the usual topology of locally uniform convergence.

Multiplicative convolution operators $T = T_\mu$ acting on $H(\mathbb{C}_*)$ can be defined as

$$Tf(z) := \langle f(z/\cdot), \mu\rangle \qquad (f \in H(\mathbb{C}_*)),$$

where μ is some analytic functional with carrier contained in $\mathbb{C}_*$.

According to Köthe-Grothendieck duality, each analytic functional as above corresponds in a unique way to a function φ holomorphic on the complement of the carrier of μ with respect to the Riemann sphere. For $\rho > 0$ we define $\tau_\rho(t) := \rho e^{it}$ ($t \in [-\pi, \pi]$), i.e. τ_ρ traverses the circle $\{|z| = \rho\}$ in the positive direction. If $f \in H(\mathbb{C}_*)$ and if L is a compact set in $\mathbb{C}_*$, then, for R is sufficiently large and r sufficiently small,

$$Tf(z) = \frac{1}{2\pi i}\left(\int_{\tau_R} - \int_{\tau_r}\right) \varphi\left(\frac{z}{\zeta}\right) \frac{f(\zeta)}{\zeta}\, d\zeta \quad (z \in L). \tag{1}$$

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Since φ is holomorphic at 0 and at ∞, we have

$$\varphi(z) = \sum_{\nu=0}^{\infty} \varphi_\nu z^\nu \tag{2}$$

near 0 and

$$\varphi(z) = \sum_{\nu=1}^{\infty} -\varphi_{-\nu} z^{-\nu} \tag{3}$$

near ∞. Denoting by $p_k : \mathbb{C}_* \to \mathbb{C}_*$ the k-th monomial, that is, $p_k(z) := z^k$ for $z \in \mathbb{C}_*$ and $k \in \mathbb{Z}$, and plugging in (2) for the first integral and (3) for the second integral on the right hand side of (1), leads to the basic property

$$Tp_k = \varphi_k \cdot p_k \qquad (k \in \mathbb{Z}) \tag{4}$$

and in particular to $(Tp_k)(1) = \varphi_k$. More generally, if f has the Laurent expansion $\sum_{\nu=-\infty}^{\infty} f_\nu z^\nu$ in $\mathbb{C}_*$, then the continuity of T implies

$$Tf(z) = \sum_{\nu=-\infty}^{\infty} \varphi_\nu f_\nu z^\nu \qquad (z \in \mathbb{C}_*).$$

We consider U to be an open subset of $\mathbb{C}_*$. Our aim is to study multiplicative convolution operators on $H(U)$ (and on appropriate subspaces of $H(U)$), which we are going to introduce now. Of importance is the concept of Cauchy and anti-Cauchy cycles.

If $L \subset U$ is compact, a cycle Γ in $U \setminus L$ is called a Cauchy cycle for L in U if

$$\mathrm{ind}_\Gamma(w) = 1 \quad (w \in L) \quad \text{ and } \quad \mathrm{ind}_\Gamma(w) = 0 \quad (w \in \mathbb{C} \setminus U)$$

(note that we always require $\mathrm{ind}_\Gamma(0) = 0$). According to [27, Theorem 13.5], for each pair (U, L) as above a Cauchy cycle exists. For basic notations and facts concerning cycles we refer to [27, Chapter 10]. If $f \in H(U)$ and if Γ is a Cauchy cycle for L in U, then Cauchy's theorem ([27, Therorem 10.35]) implies that

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{1 - z/\zeta} \frac{f(\zeta)}{\zeta} \, d\zeta \quad (z \in L).$$

Let in the sequel Ω always be open in $\mathbb{C}_*$ and so that $\Omega \cup \{0, \infty\}$ is open in $\mathbb{C}_\infty$, where $\mathbb{C}_\infty$ denotes the extended plane. In this case we say that Ω is spherically open. If, in addition, Ω is connected, we call Ω a spherical domain. Moreover, we consider a function $\varphi \in H(\Omega)$

Finally, for $A, B \subset \mathbb{C}_*$ we define $A^* := 1/(\mathbb{C}_* \setminus A)$ (with $1/\emptyset := \emptyset$) and

$$A * B := (A^* \cdot B^*)^*$$

where, as usual, $C \cdot D := \{zw : z \in C, w \in D\}$ for $C, D \subset \mathbb{C}$. The following fact plays a basic role : For an arbitrary set $S \subset \mathbb{C}_*$ we have

$$S \subset A * B \text{ if and only if } S \cdot A^* \subset B \tag{5}$$

(see [22]). Moreover, if S is compact, then $S \cdot \Omega^*$ is compact .
Let $f \in H(U)$, $L \subset \Omega * U$ compact, and $\Gamma = \Gamma_L$ a Cauchy cycle for $L \cdot \Omega^*$ in U. Then $z/\zeta \in \Omega$ for $z \in L$ and $\zeta \in \Gamma$ (note that $\zeta \notin z \cdot \Omega^*$ and thus $\zeta/z \notin \Omega^*$), and from Cauchy's theorem one can deduce that

$$(\varphi * f)(z) = (\varphi *_{\Omega,U} f)(z) := \frac{1}{2\pi i} \int_\Gamma \varphi\left(\frac{z}{\zeta}\right) \frac{f(\zeta)}{\zeta}\, d\zeta \quad (z \in L) \tag{6}$$

defines (independently of the choice of Γ) a function $\varphi * f \in H(\Omega * U)$. Moreover, the mapping

$$H(\Omega) \times H(U) \ni (\varphi, f) \mapsto \varphi * f \in H(\Omega * U)$$

is bilinear and continuous (cf. [9], [20], [22]). The definition shows that the following compatibility property is satisfied: If $\tilde{U} \supset U$ and $\tilde{\Omega} \supset \Omega$ are as above, then

$$(\varphi|_\Omega) *_{\Omega,U} (f|_U) = \varphi *_{\tilde{\Omega},\tilde{U}} f$$

for all $\varphi \in H(\tilde{\Omega})$ and $f \in H(\tilde{\Omega})$. This justifies the suppression of the open sets by writing briefly $\varphi * f$.
We emphasize a special case: If $\Omega = \mathbb{C}_* \setminus \{1\}$ then $\Omega * U = U$ for arbitrary open sets U as above and thus $\varphi * f \in H(U)$. In the case of the Cauchy kernel $\varphi = \mathbf{1}_* \in H(\mathbb{C}_* \setminus \{1\})$, i.e. $\mathbf{1}_*(z) := 1/(1-z)$, we obtain $\mathbf{1}_* * f = f$.

In the sequel we write Γ^- for the "opposite" cycle of Γ, that is, the paths constituting Γ are traversed in the opposite direction (cf. [27, p. 218]). Then for compact $L \subset \mathbb{C}_*$ the cycle Γ consisting of τ_R and τ_r^- for R sufficiently large and r sufficiently small is a Cauchy cycle for L in $U = \mathbb{C}_*$. In this case, the right hand side of (6) equals the right hand side of (1).
Let V be spherically open and $B \subset V$ closed in $\mathbb{C}_*$. We call a cycle Γ in $V \setminus B$ an anti-Cauchy cycle for B in V if

$$\mathrm{ind}_\Gamma(w) = 0 \quad (w \in B) \qquad \text{and} \qquad \mathrm{ind}_\Gamma(w) = -1 \quad (w \in \mathbb{C}_* \setminus V).$$

For each (V, B) as above (with $V \neq B$), an anti-Cauchy cycle exists, and for $z \in \Omega * U$, a cycle Γ is a Cauchy cycle for $z \cdot \Omega^*$ in U if and only if z/Γ^- is a anti-Cauchy cycle for $z \cdot U^*$ in Ω (see [22]).
If $f \in H(U)$, $L \subset \Omega * U$ compact, and Γ an anti-Cauchy cycle for $L \cdot U^*$ in Ω we define

$$(f * \varphi)(z) = (f *_{U,\Omega} \varphi)(z) := \frac{1}{2\pi i} \int_\Gamma f\left(\frac{z}{\omega}\right) \frac{\varphi(\omega)}{\omega}\, d\omega \quad (z \in L)$$

(independently of the choice of Γ). Then the substitution $\zeta = z/\omega$ leads to

$$\varphi * f = f * \varphi$$

that is, the convolution product is commutative.

Let Ω be spherically open, $\varphi \in H(\Omega)$ and $U \subset \mathbb{C}_*$ open. The main objective of this paper is the investigation of the (continuous) operator

$$T_\varphi = T_{\varphi,U} : H(U) \to H(\Omega * U), \qquad T_\varphi f := \varphi * f \qquad (f \in H(U))$$

and of natural restrictions of this operator. We write $p_{k,M} := p_k|_M$ for $M \subset \mathbb{C}_*$. From (4) and the compatibility property (with $\tilde{U} = \mathbb{C}_*$) it follows that for all open subsets U of $\mathbb{C}_*$

$$T_\varphi p_{k,U} = \varphi_k \cdot p_{k,\Omega * U} \qquad (k \in \mathbb{Z}). \tag{7}$$

If, in addition, $\mathbb{C}_* \setminus U$ has no compact components, then Runge's theorem shows that the span of the $p_{k,U}$ is dense in H(U) and thus T_φ is uniquely determined by the sequence of multipliers (φ_k).
In the special case $\Omega = \mathbb{C}_* \setminus \{1\}$ we have $T_\varphi : H(U) \to H(U)$ and thus (7) shows that the monomials $p_{k,U}$ are eigenfunctions corresponding to the eigenvalues φ_k, that is, T_φ is a multiplier on $H(U)$. In this case, the operator T_φ may be written as an infinite order Euler differential operator (cf. [10, Section 11.2]). Euler differential operators and multipliers on spaces of real analytic functions were rigorously investigated in a series of publications by Domański and Langenbruch ([5], [6], [7]), cf. also the papers [11], [12] of Ishimura.
If $U \cup \{0\}$ is open in $\mathbb{C}$ and if $f(z) = \sum_{\nu=0}^\infty f_\nu z^\nu =: f^+(z)$ on $0 < |z| < r$ for some $r > 0$ (that is, f has a removable singularity at 0), we obtain from (7), the continuity of $T_{\varphi,\{0<|z|<r\}}$ and the compatibility property that

$$(\varphi * f)(z) = \sum_{\nu=0}^\infty \varphi_\nu f_\nu z^\nu \tag{8}$$

for $z \neq 0$ near 0. This is the Hadamard multiplication theorem in a general form (see e.g. [9], [20], [22], and [10, Theorem 11.6.1], [28, Theorem 3.2] for the classcal "starlike" version). Because of this connection, we call $\varphi * f$ Hadamard convolution product of φ and f and T_φ Hadamard convolution operator. For further results on the Hadamard product and the Hadamard operator see e.g. [4], [21], [24], [25], [26].
Similarly, if $U \cup \{\infty\}$ is open in $\mathbb{C}_\infty$ and $f(z) = \sum_{\nu=1}^\infty -f_{-\nu} z^{-\nu} =: f^-(z)$ near ∞ then

$$(\varphi * f)(z) = \sum_{\nu=1}^\infty -\varphi_{-\nu} f_{-\nu} z^{-\nu} \tag{9}$$

near ∞.
We write $H^+(U)$ for the closed subspace of $H(U)$ consisting of those functions having a removable singularity at 0 (in case that $U \cup \{0\}$ is open) and $H^-(U)$ for the closed subspace of functions that vanish at ∞ (in case that $U \cup \{\infty\}$ is open). By $H^\pm(U)$ we denote the intersection of both spaces (if U is spherically open). According to our assumptions, we always have $\varphi \in H^\pm(\Omega)$.
If we put

$$M_+ := M \cup \{0\}, \quad M_- := M \cup \{\infty\}, \quad M_\pm := M \cup \{0, \infty\}$$

for $M \subset \mathbb{C}_*$, then $H^+(U)$ is in an obvious way isomorphic to $H(U_+)$, and similarly $H^-(U) \cong H(U_-)$ and $H^\pm(U) \cong H(U_\pm)$. With that, we define $T_\varphi^+ : H^+(U) \to H^+(\Omega{*}U)$, $T_\varphi^- : H^-(U) \to H^-(\Omega{*}U)$ and $T_\varphi^\pm : H^\pm(U) \to H^\pm(\Omega{*}U)$ by restriction of T_φ to the corresponding subspaces (note that, according to (8) and (9), we have $T_\varphi(H^*(U)) \subset H^+(\Omega * U)$ and $T_\varphi(H^-(U)) \subset H^-(\Omega * U)$, respectively).
The above operators are already introduced in [22], where, however, the definition of T_φ is given for subsets of the extended plane $\mathbb{C}_\infty$ instead of the punctured plane $\mathbb{C}_*$. This approach requires the distinction of a number of different cases depending on whether 0 or ∞ belong to U or not. The above approach reduces the underlying calculations considerably.

The paper is arranged as follows: In Sections 2 and 3 we consider spherical domains Ω of a special form. In the case of open sets U having simply connected components (that is, $\mathbb{C}_\infty \setminus U$ is connected) studied in Section 2, results about injectivity, denseness of the range and surjectivity for T_φ can be obtained from corresponding (known) results for additive convolution operators. In Section 3 we use Köthe-Grothendieck duality in order to describe the dual operator of T_φ. The main ingredient for the proof is an associative law for the Hadamard convolution product. Moreover, applications of duality concerning injectivity, denseness of the range and surjectivity are given, here mainly for the operators $T_\varphi^\pm$. Finally, in Section 4, we consider more general spherically open sets Ω.

2 Conjugation to additive convolution

Let $K \subset \mathbb{S} := \{w \in \mathbb{C} : |\mathrm{Im}(w)| < \pi\}$ be compact and convex. Then

$$\Omega_K := (e^K)^*$$

is spherically open. We write $\mathcal{K}$ for the set of all $K \neq \emptyset$ as above, that is, $K \subset \mathbb{S}$ convex and compact.
It is known (see [2, Section 4.1]) that there is a one-to-one correspondence between the analytic functionals with convex carrier contained in K and $H(\Omega_K)$ given by the so-called G-transform, that is,

$$G\mu(z) := \left\langle \zeta \mapsto \frac{1}{1 - ze^\zeta}\, ,\, \mu \right\rangle \qquad (z \in \Omega_K)$$

for μ an analytic functional with convex carrier in K. Moreover, for $\varphi \in H(\Omega_K)$, the *Mellin transform* $M\varphi$ of φ is given by

$$M\varphi(\alpha) := \frac{1}{2\pi i} \int_{\Gamma^-} \frac{\varphi(\zeta)}{\zeta^{\alpha+1}}\, d\zeta \qquad (\alpha \in \mathbb{C})\, ,$$

where Γ is a Cauchy cycle for e^{-K} in $\mathbb{C}_- := \mathbb{C} \setminus (-\infty, 0]$, and $\zeta^c := \exp(c \log \zeta)$ with the principal branch of the logarithm. The *Mellin transformation* $M : H(\Omega_K) \to \mathrm{Exp}(K)$ turns out to be an isomorphism between $H(\Omega_K)$ and $\mathrm{Exp}(K)$,

the space of entire functions of exponential type having conjugate indicator diagram contained in K (see e.g. [2, p. 82, p. 266]). In the sequel we often write $\Phi := M\varphi$ for brevity. Again referring to [2, Section 4.1], we have $M = \mathcal{F} \circ G^{-1}$, where $\mathcal{F}$ denotes the Fourier-Borel transformation.
The situation in the special case $K = \{0\}$ is known as the Wigert-Leau Theorem. In this case $\Omega_K = \mathbb{C}_* \setminus \{1\}$ and $\Phi = M\varphi$ is of exponential type zero. Moreover, we then have $\Omega_K * U = U$ and T_φ is an Euler differential operator of the form

$$T_\varphi f = \Phi(\vartheta) f := \sum_{k=0}^{\infty} \Phi_k \vartheta^k f \qquad (f \in H(U))$$

where $(\vartheta f)(z) := z f'(z)$ and $\Phi(\alpha) = \sum_{k=0}^{\infty} \Phi_k \alpha^k$ (see, e.g. [2, pp. 71, pp. 419], [10, Section 11.2], [22]). Conversely, since M is bijective, for each Φ of exponential type zero the operator $\Phi(\vartheta)$ is of the form $T_{M^{-1}\Phi}$. For $\Phi(\alpha) = \alpha$ we get the Koebe function $\kappa(z) := z/(1-z)^2 = \sum_{\nu=0}^{\infty} \nu z^\nu$ and $T_\kappa f = \vartheta f$.

In this section we consider open sets $U \subset \mathbb{C}_*$ with simply connected components. Here, by appropriate exp/log substitution, the multiplicative convolution operators can be conjugated to additive convolution operators. We frequently use the fact that $\Omega * U$ has simply connected components if U has simply connected components (this follows from $(\Omega * U)^* = \Omega^* U^* = \bigcup_{w \in \Omega^*} w \cdot U^*$, the connectedness of $\mathbb{C}_\infty \setminus w \cdot U^*$, and $0 \in \mathbb{C}_\infty \setminus w \cdot U^*$ for $w \neq 0$). Moreover, we write $U_\delta(z_0) := \{z : |z - z_0| < \delta\}$ for $\delta > 0$ and $z_0 \in \mathbb{C}$. Again, by log we denote the principal brach of the logarithm on the cut plane $\mathbb{C}_-$.

2.1 Lemma. *Let $U \subset \mathbb{C}_*$ be open with simply connected components and $\Omega = \Omega_K$ for some $K \in \mathcal{K}$. Then for each logarithm $\log_U$ on U there exists a logarithm $\log_{\Omega * U}$ on $\Omega * U$ such that*

$$\log_U\left(\frac{z}{\zeta}\right) = \log_{\Omega * U}(z) - \log(\zeta) \tag{10}$$

for $z \in \Omega * U$ and ζ belonging to a sufficiently small neighborhood of e^{-K}.

Proof. We fix a branch of the logarithm on each component of U and denote the resulting holomorphic function on U by $\log_U$. Our aim is to show that branches of the logarithm on the components of $\Omega * U$ can be chosen in such a way that the asserted identity holds.
It is clear that there exists a number $a \in K$ such that the set $K - a$ contains the origin (if K itself contains the origin we choose $a = 0$). This implies that $1 \notin e^a \cdot \Omega$ and therefore $e^a \cdot (\Omega * U) \subset U$. Especially, every component of $e^a \cdot (\Omega * U)$ is a subset of a component of U. Therefore it is meaningful to set $\log_{e^a \cdot (\Omega * U)}(e^a z) := \log_U(e^a z)$ $(z \in \Omega * U)$.
Obviously, every branch of the logarithm on $\Omega * U$ fulfills the following equation for all $z \in \Omega * U$:

$$\log_{\Omega * U}(z) = \log_U(e^a z) - a + 2\pi i k(z)$$

for some $k(z) \in \mathbb{Z}$. The map

$$\Omega * U \ni z \mapsto \log_{\Omega * U}(z) - (\log_U(e^a z) - a) \in \mathbb{C}$$

is continuous and its range is a discrete subset of $\mathbb{C}$. Therefore it must be constant on every component of $\Omega * U$. Hence, the branch of the logarithm on every component of $\Omega * U$ can be chosen so that

$$\log_{\Omega * U}(z) = \log_U(e^a z) - a \quad (z \in \Omega * U) . \tag{11}$$

Let now $z \in \Omega * U$ be given. The set $z \cdot U^*$ is a compact subset of the open set Ω (see (5)) and therefore we can find a number $\delta_1 = \delta_1(z) > 0$ such that $(e^{-K} + U_{\delta_1}(0)) \cap z \cdot U^* = \emptyset$. On the other hand, e^{-K} is a compact subset of the open set $\mathbb{C}_-$ and therefore we can find a number $\delta_2 > 0$ such that $e^{-K} + U_{\delta_2}(0) \subset \mathbb{C}_-$. We set $\delta := \min\{\delta_1, \delta_2\}$ and $V_\delta := e^{-K} + U_\delta(0)$.
In order to prove (10) we first of all note that the left-hand side of (10) is defined since $z/\zeta \in U$ for all $\zeta \in V_\delta$. Indeed, assuming the existence of a number $w \in U^C$ with $z/\zeta = w$ would imply $z \cdot U^* \ni z/w = \zeta \in V_\delta$ which contradicts the choice of δ.
Obviously we have

$$g_z(\zeta) := \log_U(\frac{z}{\zeta}) - (\log_{\Omega * U}(z) - \log(\zeta)) = 2k_z(\zeta)\pi i \quad (\zeta \in V_\delta)$$

for some $k_z(\zeta) \in \mathbb{Z}$. The same argument as above yields that g_z is constant on V_δ (note that V_δ is connected). Inserting $\zeta_0 = e^{-a} \in V_\delta$ implies (with (11) and noting that $a \in K$ and therefore $\log(e^{-a}) = -a$)

$$g_z(\zeta) = g_z(\zeta_0) = \log_U(e^a z) - (\log_U(e^a z) - a - \log(e^{-a})) = 0 \quad (\zeta \in V_\delta).$$

This completes the proof of the asserted equation. $\square$

Let O be an open subset of $\mathbb{C}$ and let μ be an analytic functional with convex carrier in K. We consider additive convolution operators $S = S_\mu : H(O+K) \to H(O)$ of the form

$$Sg(w) := \langle h(w + \cdot), \mu \rangle \qquad (h \in H(O + K),\, w \in O)$$

(see, e.g. [2], [16]). If $U \subset \mathbb{C}_-$ has simply connected components and if logarithms on U and $\Omega * U$ are fixed as in Lemma 2.1, then (5) implies that

$$(\Omega * U)_{\log} + K \subset U_{\log},$$

where we write $U_{\log} := \log_U(U)$ and $(\Omega * U)_{\log} := \log_{\Omega * U}(\Omega * U)$ for brevity.
For $\varphi = G\mu$ we define $S_\varphi : H(U_{\log}) \to H((\Omega * U)_{\log})$ as the restriction of the additive convolution operator S to $H(U_{\log})$. Then we have

$$S_\varphi h(w) = \frac{1}{2\pi i} \int_\Gamma \varphi(\zeta) h(w - \log(\zeta)) \frac{d\zeta}{\zeta} \quad (h \in H(U_{\log}), w \in (\Omega * U)_{\log}),$$

where Γ is an anti-Cauchy cycle for $e^w \cdot U^*$ in $\Omega \cap \mathbb{C}_-$ (cf. [2, Section 4.1] and note that e^w/Γ^- is a Cauchy cycle for $e^w \cdot \Omega^*$ in U and therefore Γ^- is a Cauchy cycle for e^{-K} in $(e^w/U) \cap \mathbb{C}_-$). Now, according to Lemma 2.1 we have

$$(S_\varphi h) \circ \log_{\Omega * U} = T_\varphi(h \circ \log_U) \quad (h \in H(U_{\log})) \tag{12}$$

and the following diagram commutes:

$$\begin{tikzcd} H(U_{\log}) \arrow[r, "S_\varphi"] \arrow[d, shift left, "\circ \log_U"'] & H((\Omega * U)_{\log}) \arrow[d, shift left, "\circ \log_{\Omega * U}"'] \\ H(U) \arrow[r, "T_\varphi"'] \arrow[u, shift left, "\circ \exp|_{U_{\log}}"] & H(\Omega * U) \arrow[u, shift left, "\circ \exp|_{(\Omega * U)_{\log}}"] \end{tikzcd}$$

With that it is possible to transfer (known) results for additive convolution operators to our multiplicative Hadamard operators.

2.2 Theorem. *Let $U \subset \mathbb{C}_*$ be open with simply connected components and $\Omega = \Omega_K$ for some $K \in \mathcal{K}$. If $\varphi \in H(\Omega) \setminus \{0\}$ then T_φ has dense range.*

Proof. It is known that for $\mu \neq 0$ the additive convolution operator S : $H(\mathbb{C}) \to H(\mathbb{C})$ is surjective (see, e.g. [2, Proposition 1.5.8]). Since $\log_{\Omega * U}$ is one-to-one from $\Omega * U$ to $(\Omega * U)_{\log}$, the open set $(\Omega * U)_{\log}$ has simply simply connected components. According to Runge's theorem, the operator $S_\varphi : H(U_{\log}) \to H((\Omega * U)_{\log})$ has dense range. But then also T_φ has dense range. $\square$

In the sequel we use the abbreviations $K_\delta := i[-\pi\delta, \pi\delta]$ and $B_\delta := e^{K_\delta}$ for $\delta \geq 0$. Moreover, we put $\arg z := \operatorname{Im} \log z \in (-\pi, \pi]$.

2.3 Example. 1. We consider the simple but illustrating example $\Omega = \mathbb{C}_* \setminus \{\pm i\}$,

$$\varphi(z) = \frac{1}{1+z^2} = \frac{1}{2}\left(\frac{1}{1-e^{i\pi/2}z} + \frac{1}{1-e^{-i\pi/2}z}\right) \quad (z \in \Omega).$$

If $U = \mathbb{C}_-$ then $\Omega * U = \{z : \operatorname{Re}(z) \neq 0\}$. Since φ is even, by definition the same holds for $\varphi * f$ for all $f \in H(U)$ (take $L = \{\pm z\}$ in (6)). Thus, $\operatorname{im}(T_\varphi)$ is not dense in $H(\Omega * U)$. On the other hand, since $M\mathbf{1}_*(e^\beta \cdot)(\alpha) = e^{\beta\alpha}$ for $\alpha, \beta \in \mathbb{C}$, we see that

$$\Phi(\alpha) = M\varphi(\alpha) = \cos(\alpha\pi/2) \qquad (\alpha \in \mathbb{C}).$$

The (conjugate) indicator diagram of Φ is the line segment $K_{1/2}$ on the imaginary axis. Therefore, $\Omega_K * U = \{z : \operatorname{Re}(z) > 0\}$ is the right half plane. According to Theorem 2.2, $\operatorname{im}(T_{\varphi|_{\Omega_K}})$ is dense in $H(\Omega_K * U)$.
2. Let $\{\alpha_n : n \in \mathbb{N}\}$ be an increasing sequence of positive integers with $\lim_{n\to\infty} n/\alpha_n = \delta < 1$ and

$$\Phi(\alpha) := \prod_{n=1}^{\infty}\left(1 - \frac{\alpha^2}{\alpha_n^2}\right) \qquad (\alpha \in \mathbb{C}).$$

Then Φ is of exponential type with (conjugate) indicator diagram K_δ (see, e.g. [17, p. 205]). Hence $\varphi = M^{-1}\Phi$ is holomorphic in $\Omega = B_\delta^*$.
If $U = \mathbb{C}_-$ then $\Omega * U = \{z : |\arg(z)| < \pi(1-\delta)\}$ and Theorem 2.2 yields that the operator T_φ has dense range. The example in 1. is embedded as the special case $\alpha_n = 2n+1$.

For an open set $U \subset \mathbb{C}_*$ with simply connected components let

$$p_{\alpha,U}(z) := e^{\alpha \log_U z}$$

for $z \in U$ and $\alpha \in \mathbb{C}$, where $\log_U$ is some logarithm on U. If $\log_{\Omega * U}$ is a logarithm on $\Omega * U$ according to Lemma 2.1, then for $\alpha \in \mathbb{C}$

$$T_\varphi p_{\alpha,U} = (S_\varphi \exp(\alpha\,\cdot)) \circ \log_{\Omega*U} = \Phi(\alpha) p_{\alpha,\Omega*U}. \tag{13}$$

Comparing with (7) it is seen that, in particular,

$$\varphi_k = \Phi(k) \quad (k \in \mathbb{Z}). \tag{14}$$

As a consequence we obtain

2.4 Theorem. *If $U \subset \mathbb{C}_*$ is a simply connected domain and if $\varphi \in H(\Omega)$, where $\Omega = \Omega_K$ for some $K \in \mathcal{K}$, then the following are equivalent:*

1. *T_φ is injective,*
2. *Φ has no zeros,*
3. *φ is a nonzero multiple of $\mathbf{1}_*(e^\beta \cdot)$ for some $\beta \in K$.*

Proof. (13) yields that 1. implies 2.
In order to show that 2. implies 3. we assume that $\Phi \in \mathrm{Exp}(K)$ has no zeros. Then according to the Hadamard factorization theorem (see e.g. [3, Th. 2.7.1]), there are numbers $\alpha, \beta \in \mathbb{C}$ such that $\Phi(z) = \exp(\beta z + \alpha)$ $(z \in \mathbb{C})$. In order that the condition $\Phi \in \mathrm{Exp}(K)$ is satisfied, β must belong to the set K. Setting $\lambda := e^\alpha \neq 0$, the power series expansion of φ about zero yields that

$$\varphi(z) = \frac{\lambda}{1 - e^\beta z} \qquad (z \in \Omega)\,.$$

To prove that 3. implies 1. we examine how T_φ acts on a function $f \in H(U)$. For all $z \in \Omega * U$ we obtain

$$T_\varphi f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\lambda}{1 - e^\beta z/\zeta} f(\zeta) \frac{d\zeta}{\zeta} = \lambda f(e^\beta z)$$

where Γ is a Cauchy cycle for $z \cdot e^K$ in U and the last identity follows from the Cauchy integral formula. Since U is connected, the operator T_φ is injective. $\square$

For the special case $K = \{0\}$ we obtain from Theorems 2.2 and 2.4.

2.5 Corollary. Let $U \subset \mathbb{C}_*$ be open with simply connected components and let $\Phi \neq 0$ be an entire function of exponential type 0. Then

1. $\Phi(\vartheta)$ has dense range.
2. $\Phi(\vartheta)$ is injective if and only if $\Phi(\vartheta)$ is a nonzero multiple of the identity on $H(U)$.

2.6 Remark. The situation changes drastically if U does not have simply connected components: If $U = \mathbb{C}_* (= \Omega * U)$ and if Φ has a zero at some integer, then Laurent expansion and (4) show that $\Phi(\vartheta) : H(U) \to H(U)$ has no longer dense range. For instance, if $\Phi(\alpha) = \alpha$, that is, $T_\varphi f = \vartheta f$, then the function $g = 1$ does not belong to the closure of $\mathrm{im}(T_\varphi)$.

We now turn towards the question under which conditions the operator T_φ is even surjective.
Let $\Phi \neq 0$ be an entire function of exponential type. Then Φ is said to be of *completely regular growth* if there exists a set E of relative zero (Lebesgue) measure such that $\lim_{E \not\ni r \to \infty} r^{-1} \log |\Phi(re^{it})|$ exists uniformly in $t \in [-\pi, \pi]$ (for this notion, see [17]; cf. [2, p. 88]). Moreover, by $K(\Phi)$ we denote the conjugate indicator diagram of Φ (see e.g. [3] or [17]). Then we have

2.7 Theorem. *Let $K \in \mathcal{K}$ and $\varphi \in H(\Omega)$, where $\Omega = \Omega_K$. Moreover, we suppose that $\Phi = M\varphi$ is an entire function of completely regular growth with $K(\Phi) = K$. If $W \subset \mathbb{C}$ is open and convex with $W + K \subset \mathbb{S}$ and $U := e^{W+K}$, then $\Omega * U = e^W$ and T_φ is surjective.*

Proof.
1. We show that $\Omega * U = e^W$. Since $U = e^{W+K} = e^W \cdot \Omega^*$, (5) implies that $\Omega * U$ is a superset of e^W. To obtain the reverse inclusion, we show $(e^W)^* \subset (\Omega * U)^*$. We have

$$(\Omega * U)^* = \Omega^* \cdot U^* = e^K \cdot (e^{W+K})^* = e^K \cdot e^{-(\mathbb{C} \setminus O)}$$

where $O := \bigcup_{k \in \mathbb{Z}} (W + K + 2k\pi i)$ (note that the sets $W + K + 2k\pi i$ $(k \in \mathbb{Z})$ are pairwise disjoint).
Now let $z \in (e^W)^*$. Then there is a point $v \in \overline{\mathbb{S}} \setminus W$ such that for all $u \in K$ we have

$$z = e^{-v} = e^u \cdot e^{-(v+u)}.$$

If $u \in K$ can be chosen in such a way that $v + u \notin O$ we are done.
Assume that this is not the case, i.e. $v + K \subset O$. Since $v + K$ is connected, it has to lie entirely in one component of O and that component shall without loss of generality be the set $W + K$ itself. Since W is convex, without loss of generality we can choose an exhaustion $(L_n)_{n \in \mathbb{N}}$ of W (i.e. (L_n) has the properties as in [27, Theorem 13.3]) consisting of convex sets. Then $(L_n + K)$ is an exhaustion of $W + K$ consisting of convex (and compact) sets. Since $v + K$ is compact, there is an integer n_0 such that $v + K \subset L_{n_0} + K$. Moreover, since in the latter inclusion all occurring sets are compact and convex, we can deduce $v \in L_{n_0} \subset W$ (which

follows from the above properties of the support function). This contradicts the choice of v.
2. Without loss of generality we may assume that $0 \in K$ and thus $\Omega * U \subset U$ (Indeed: If $a \in K$, then $\tilde{\Phi} = \Phi \exp(-a\,\cdot)$ has indicator diagram $K - a$ with $0 \in K - a$. Then $\tilde{\varphi} = M^{-1}\tilde{\Phi}$ and φ differ only by a multipicative scaling of the variable.)
Let log be the principal branch of the logarithm. If we choose $\log_{\Omega * U} = \log_U = \log$, then the functional equation from Lemma 2.1 is satisfied. From 1. we have

$$\log(\Omega * U) + K = W + K = \log(U).$$

It is known that under the above assumptions, the additive convolution operator $S_\varphi : H(W + K) \to H(W)$ is surjective. The corresponding result is found e.g. in [2, 1.5.12] and [16, Theorem 6.1]), where it has to be noted that the Fourier-Borel transform (or the Laplace transform transform in the language of [16]) of the analytic functional μ coincides with the Mellin-transform Φ of $\varphi = G\mu$ and that $K(\Phi)$ equals the convex carrier of μ. But then, according to (12), also T_φ is surjective. □

2.8 Example. We consider again the situation in Example 2.3.2. There we stated that T_φ has dense range. Actually the function Φ is of completely regular growth with $K(\Phi) = K_\delta$ (see e.g. [17], p. 205). Hence, Theorem 2.7 (with $W = \{w : |\mathrm{Im}(w)| < \pi(1 - \delta\}$) yields that the operator T_φ is even surjective.

Since each function of exponential type zero is of completely regular growth (see e.g. [2, p. 90], [17, p. 158]), we obtain from Theorem 2.7:

2.9 Corollary. Let $\Phi \neq 0$ be of exponential type 0 and let $U \subset \mathbb{C}_-$ be a simply connected domain so that $\log(U)$ is convex. Then $\Phi(\vartheta)$ is surjective.

2.10 Remark. Similar results for corresponding classical differential operators of infinite order $\Phi(D) : H(O) \to H(O)$,

$$\Phi(D)h = \sum_{k=0}^{\infty} \Phi_k D^k h \qquad (h \in H(O)),$$

where $Dh := h'$ and $O \subset \mathbb{C}$ an open set, are well-known (see e.g. [2, Theorem 6.4.4]). As a special case of (12), for $U \subset \mathbb{C}_*$ with simply connected components, the following diagram commutes:

$$\begin{tikzcd}
H(U_{\log}) \arrow[r, "\Phi(D)"] \arrow[d, leftrightarrow, "\circ \exp|_{U_{\log}}"', "\circ \log_U"] & H(U_{\log}) \arrow[d, leftrightarrow, "\circ \exp|_{U_{\log}}"', "\circ \log_U"] \\
H(U) \arrow[r, "\Phi(\vartheta)"'] & H(U)
\end{tikzcd}$$

This shows that results for the classical differential operators on $H(U_{\log})$ and the corresponding Euler differential operators on $H(U)$ turn out to be equivalent. In

particular, since finite order differential operators are surjective as operators on $H(O)$ for all open sets $O \subset \mathbb{C}$ having simply connected components, we obtain that for $U \subset \mathbb{C}_*$ having simply connected components and $\varphi \in H(\mathbb{C}_* \setminus \{1\})$ with $\Phi = M\varphi$ a polynomial, the operator $T_\varphi = \Phi(\vartheta) : H(U) \to H(U)$ is surjective as well. This shows that for a polynomial Φ the convexity of $U_{\log}$ is not necessary for surjectivity of $\Phi(\vartheta)$.
On the other hand, for transcendental Φ convexity of O turns out to be necessary for surjectivity of $\Phi(D)$ (see [15]). Thus, for transcendental Φ, convexity of $U_{\log}$ is necessary for surjectivity of $\Phi(\vartheta)$.

2.11 Remark. In [8], Frerick provides characterizations of the surjectivity of the one-sided operators $\Phi(\vartheta)^+ : H^+(U) \to H^+(U)$:

1. $\Phi(\vartheta)^+$ is surjective, for all U so that U_+ is simply connected, if and only if $Z(\Phi) \cap \mathbb{N}_0 = \emptyset$ and Φ is a polynomial or
$$\lim_{\alpha \to \infty,\, \alpha \in Z(\Phi)} \alpha/|\alpha| = -1.$$

2. $\Phi(\vartheta)^+$ is surjective, for all U so that U_+ is starlike with respect to the origin, if and only if $Z(\Phi) \cap \mathbb{N}_0 = \emptyset$ and Φ is a polynomial or
$$\limsup_{\alpha \to \infty,\, \alpha \in Z(\Phi)} \operatorname{Re}(\alpha/|\alpha|) \leq 0.$$

3 Transpose of T_φ and applications

In order to describe the transpose of T_φ it is important to provide an associative law for the Hadamard convolution product.
Let $U \subset \mathbb{C}_*$ be open and $K \subset U$ compact. The hull $h_U(K)$ of K with respect to U is defined as the union of K and all relatively compact components of $U \setminus K$. This implies that each component of $(h_U(K))^*$ meets a component of U^*. For a cycle Γ we denote by $|\Gamma|$ the trace of Γ, i.e. the union of the images of the closed paths constituting Γ.

3.1 Theorem. *Let $U \subset \mathbb{C}_*$ be open and $\Omega, V \subset \mathbb{C}_*$ spherically open. If $f \in H(U)$, $g \in H^\pm(V)$, and $\varphi \in H^\pm(\Omega)$ then*
$$g * (\varphi * f) = (g * \varphi) * f.$$

Proof. Let $w \in V * (\Omega * U)$. We choose Γ_1 to be an anti-Cauchy cycle for $w \cdot (\Omega * U)^*$ in V. Then
$$\operatorname{ind}_{\Gamma_1}(z) = 0 \;\; (z \in w \cdot (\Omega * U)^*), \;\; \operatorname{ind}_{\Gamma_1}(z) = -1 \;\; (z \in \mathbb{C}_* \setminus V) \tag{15}$$
and we obtain additionally
$$\operatorname{ind}_{\Gamma_1}(z) = 0 \;\; \left(z \in (h_{(\Omega * U)/w}(1/|\Gamma_1|))^*\right). \tag{16}$$

Indeed: Note that $((w\cdot(\Omega*U)^*)^* = (\Omega*U)/w$. Since $|\Gamma_1|$ and $w\cdot(\Omega*U)^*$ are disjoint and thus $1/|\Gamma_1| \subset (\Omega*U)/w$, by the properties of the hull mentioned above it follows that $|\Gamma_1| \cap (h_{(\Omega*U)/w}(1/|\Gamma_1|))^* = \emptyset$ and that each component of $(h_{(\Omega*U)/w}(1/|\Gamma_1|))^*$ meets a component of $w\cdot(\Omega*U)^*$. Therefore (16) is a direct consequence of (15). We obtain

$$(g*(\varphi*f))(w) = ((\varphi*f)*g)(w) = \frac{1}{2\pi i}\int_{\Gamma_1}(\varphi*f)(\frac{w}{t})g(t)\frac{dt}{t}.$$

Now we choose Γ_2 to be a Cauchy cycle for $(w/|\Gamma_1|)\cdot\Omega^*$ in U. Actually, we impose a stronger condition and require Γ_2 to be a Cauchy cycle for $w\cdot h_{(\Omega*U)/w}(1/|\Gamma_1|)\cdot\Omega^*$ in U. This is possible since the choice of Γ_1 ensures that $|\Gamma_1|\cap w\cdot(\Omega*U)^* = \emptyset$. Hence, $(1/|\Gamma_1|)$ and consequently $h_{(\Omega*U)/w}(1/|\Gamma_1|)$ is a compact subset of $(\Omega*U)/w$. This, in turn, implies that $w\cdot h_{(\Omega*U)/w}(1/|\Gamma_1|)\cdot\Omega^*$ is a compact subset of U (see (5)).
We remark that the index property for Γ_1 implies

$$V^* \subset h_{(\Omega*U)/w}(1/|\Gamma_1|)$$

and therefore

$$(w\cdot(1/|\Gamma_1|)\cdot\Omega^*)\cup(w\cdot V^*\cdot\Omega^*) \subset w\cdot h_{(\Omega*U)/w}(1/|\Gamma_1|)\cdot\Omega^*\,. \tag{17}$$

This yields

$$(\varphi*f)(\frac{w}{t}) = \frac{1}{2\pi i}\int_{\Gamma_2}\varphi(\frac{w}{t\zeta})f(\zeta)\frac{d\zeta}{\zeta} \quad (t\in\Gamma_1)$$

and thus

$$\begin{aligned}(g*(\varphi*f))(w) &= \frac{1}{2\pi i}\int_{\Gamma_1}\frac{g(t)}{t}\frac{1}{2\pi i}\int_{\Gamma_2}\varphi(\frac{w}{t\zeta})f(\zeta)\frac{d\zeta}{\zeta}dt\\ &= \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\zeta)}{\zeta}\frac{1}{2\pi i}\int_{\Gamma_1}\varphi(\frac{w}{\zeta t})\frac{g(t)}{t}dt\,d\zeta\,.\end{aligned}$$

Now, we proceed in the opposite direction by first noting that

$$\frac{1}{2\pi i}\int_{\Gamma_1}\varphi(\frac{w}{\zeta t})\frac{g(t)}{t}\,dt = (\varphi*g)(\frac{w}{\zeta}).$$

Indeed, Γ_1 should be an anti-Cauchy cycle for $(w/|\Gamma_2|)\cdot\Omega^*$ in V. We are going to check that now:

1. $|\Gamma_1|\subset V$ according to the choice of Γ_1.

2. We have
$$w\cdot h_{(\Omega*U)/w}(1/|\Gamma_1|)\cap(|\Gamma_2|\cdot(\mathbb{C}_*\setminus\Omega)) = \emptyset\,. \tag{18}$$
Otherwise, this would contradict the fact that $|\Gamma_2|\cap w\cdot h_{(\Omega*U)/w}(1/|\Gamma_1|)\cdot\Omega^* = \emptyset$. A first consequence is that $|\Gamma_1|\cap(w/|\Gamma_2|)\cdot\Omega^* = \emptyset$.

3. A second consequence of (18) is that $(w/|\Gamma_2|)\cdot\Omega^* \subset (h_{(\Omega * U)/w}(1/|\Gamma_1|))^*$ and therefore (see (16) and (15))

$$\begin{aligned} \mathrm{ind}_{\Gamma_1}(z) = &\quad 0 \qquad (z \in (w/|\Gamma_2|)\cdot\Omega^*), \\ \mathrm{ind}_{\Gamma_1}(z) = &\quad -1 \qquad (z \in \mathbb{C}_* \setminus V)\,. \end{aligned}$$

Finally, we have

1. $|\Gamma_2| \subset U$ according to the choice of Γ_2.
2. $|\Gamma_2| \cap w\cdot(\Omega * V)^* = |\Gamma_2| \cap w \cdot \Omega^* \cdot V^* = \emptyset$ due to (17) and the choice of Γ_2.
3. A second consequence of (17) and the choice of Γ_2 is

$$\begin{aligned} \mathrm{ind}_{\Gamma_2}(z) = &\quad 1 \qquad (z \in w\cdot\Omega^*\cdot V^*)\,, \\ \mathrm{ind}_{\Gamma_2}(z) = &\quad 0 \qquad (z \in \mathbb{C} \setminus U)\,. \end{aligned}$$

Hence, Γ_2 is as a Cauchy cycle for $w\cdot(\Omega * V)^*$ in U. Therefore, we obtain

$$\frac{1}{2\pi i}\int_{\Gamma_2} (\varphi * g)(\frac{w}{\zeta}) f(\zeta)\frac{d\zeta}{\zeta} = ((\varphi * g) * f)(w) = ((g * \varphi) * f)(w).$$

□

For a closed set $B \subset \mathbb{C}_\infty$ we define

$$H(B) := \{[(g,D)]_B : D \supset B \text{ open }, \, g \in H(D)\}$$

(germs of holomorphic functions on B) with the inductive topology corresponding to the restriction maps

$$\left(j_D : H^\infty(D) \to H(B)\right)_{D \supset B\,\mathrm{open}}$$

(see, e.g. [19, p. 292]). If $B \subset \mathbb{C}_*$ is closed, then also $B_\pm \subset \mathbb{C}_\infty$ is closed. By identifying $H(D_\pm)$ and $H^\pm(D)$ for D spherically open, we get

$$H(B_\pm) = \{[(g,D)]_{B_\pm} : D \supset B \text{ spherically open }, \, g \in H^\pm(D)\}\,.$$

3.2 Remark. (Köthe duality) With the above notations, the well known representation of the dual space $H(U)'$ given by Köthe can be formulated in terms of the convolution product (cf. [9]):
Let $U \subset \mathbb{C}_*$ be open. Then to each $u \in H(U)'$ there corresponds a unique germ $[(g,D)]_{(U^*)_\pm} \in H((U^*)_\pm)$ such that

$$u(f) = (g * f)(1) \qquad (f \in H(U))\,.$$

In the sequel we identify u and $[(g,D)]_{(U^*)_\pm}$ and write also g for short.

Note for the following that $(\Omega * U)^* = \Omega^* U^*$.

3.3 Theorem. *Let $U \subset \mathbb{C}_*$ open. Then $T'_\varphi : H((\Omega^* U^*)_\pm) \to H((U^*)_\pm)$ is given by*

$$T'_\varphi[(g, V)]_{(\Omega^* U^*)_\pm} = [(g * \varphi, V * \Omega)]_{(U^*)_\pm}$$

or, briefly, $T'_\varphi g = g * \varphi (= \varphi * g)$.

Proof. For an open superset D of $(\Omega * U)^*$ we have $D^* \subset \Omega * U$, and (5) yields $(\Omega * D)^* = \Omega^* \cdot D^* \subset U$ and therefore $\Omega * D \supset U^*$. Hence $[(\varphi * g, \Omega * V)]_{(U^*)_\pm}$ belongs to the space $H((U^*)_\pm)$. Moreover, $[(g * \varphi, V * \Omega)]_{(U^*)_\pm}$ is independent of the choice of the representative (g, V).
We have to check that the unique germ corresponding to $T'_\varphi g \in H(U)'$ is given by $[(g * \varphi, V * \Omega)]_{(U^*)_\pm} \in H((U^*)_\pm)$. We apply the associative law for the Hadamard convolution product and obtain, for all $f \in H(U)$,

$$T'_\varphi g(f) = g(T_\varphi f) = (g * (\varphi * f))(1) = ((g * \varphi) * f)(1) = (g * \varphi)(f).$$

□

3.4 Remark. According to the Hahn-Banach theorem, for appropriate U as in the introduction we have (in the sense of Köthe-Grthendieck duality)

$$H^+(U)' = H((U^*)_+), \quad H^-(U)' = H((U^*)_-), \quad H^\pm(U)' = H(U^*)$$

and from that it follows that $T^+_\varphi : H((\Omega^* U^*)_+) \to H((U^*)_+)$ is given by

$$(T^+_\varphi)'[(g, V)]_{(\Omega^* U^*)_+} = [(g * \varphi, V * \Omega)]_{(U^*)_+}$$

and similarly for $(T^-_\varphi)' : H((\Omega^* U^*)_-) \to H((U^*)_-)$ and $(T^\pm_\varphi)' : H(\Omega^* U^*) \to H(U^*)$.

In the sequel we derive further results concerning injectivity and denseness of the range via duality. We use the fact that a continuous linear operator between locally convex spaces has dense range if and only if the transpose is injective.
As a dual version of Theorems 2.4 and 2.2 we obtain (note that U is spherically open if $\Omega * U$ is spherically open)

3.5 Theorem. *Let $\varphi \in H(\Omega)$, where $\Omega = \Omega_K$ for some $K \in \mathcal{K}$, and suppose that $\Omega * U$ is a spherical domain.*

1. *If U is a (spherical) domain and if $\varphi \neq 0$, then $T^\pm_\varphi$ is injective,*
2. *$T^\pm_\varphi$ has dense range if and only if $\Omega * U = e^{-\beta} U$ and φ is a nonzero multiple of $\mathbf{1}_*(e^\beta \cdot)$ for some $\beta \in K$.*

Proof.
1. Note that since $\Omega * U$ is a spherical domain there exists a closed and connected set $L \subset \Omega * U$ with $0, \infty$ belonging to the interior of $L_\pm \subset \mathbb{C}_\infty$. We set $W := L^*$ and obtain an open set in $\mathbb{C}_*$ having connected complement. Furthermore, we have $W^* = L \subset \Omega * U$ and (5) yields that $(\Omega * W)^* = W^* \cdot \Omega^*$ is a compact subset of U.

Theorem 2.2 shows that the operator $T_{\varphi,W} : H(W) \to H(\Omega * W)$ has dense range. Hence

$$T'_{\varphi,W} : H((\Omega^* W^*)_\pm) \to H((W^*)_\pm)$$

is injective.
Let now $f \in \ker(T_\varphi^\pm)$ be given. Then $[(f,U)]_{(\Omega^* W^*)_\pm} \in H((\Omega^* W^*)_\pm)$ and

$$T'_{\varphi,W}[(f,U)]_{(\Omega^* W^*)_\pm} = [(\varphi * f, \Omega * U)]_{(W^*)_\pm} = [0]_{(W^*)_\pm}.$$

Hence, $[(f,U)]_{(\Omega^* W^*)_\pm} = [0]_{(\Omega^* W^*)_\pm}$ which means that f vanishes in an open neighbourhood O of $\Omega^* W^*$. Since $O \cap U \neq \emptyset$ and since U is connected, f vanishes on U.
2. If K is so that $\Omega * U = e^{-\beta} U$ and $\varphi = \lambda \mathbf{1}_*(e^\beta \cdot)$ then $T_\varphi^\pm f = \lambda f(e^\beta \cdot)$ (see the proof of Theorem 2.4). In this case, $T_\varphi^\pm : H^\pm(U) \to H^\pm(e^{-\beta} U)$ obviously has dense range (actually, T_φ is surjective).
Conversely, we suppose that $T_\varphi^\pm$ has dense range. Then $(T_\varphi^\pm)' : H(\Omega^* U^*) \to H(U^*)$ is injective. We can choose L as in part 1. of the proof so large that L has no holes lying in $\Omega * U$. Then $W = L^*$ is so that each component contains a point of $\Omega^* U^*$. This implies that also $T_{\varphi,W} : H(W) \to (\Omega * W)$ is injective. Theorem 2.4 shows that φ has the desired form. Moreover, then K has be so that $\Omega * U = e^{-\beta} U$ since otherwise $T_\varphi^\pm$ cannot have dense range. $\square$

3.6 Example. We consider one more time the situation in Example 2.3.2. If $U = B_\eta^*$ for some $\eta < 1 - \delta$, then $\Omega * U = B_{\delta+\eta}^*$. According to Theorem 3.5, the operator $T_\varphi^\pm$ is injective.

3.7 Corollary. Let $U \subset \mathbb{C}_*$ be a spherical domain and let $\Phi \neq 0$. Then

1. $\Phi(\vartheta)^\pm : H^\pm(U) \to H^\pm(U)$ injective.
2. $\Phi(\vartheta)^\pm : H^\pm(U) \to H^\pm(U)$ has dense range if and only if it is a nonzero multiple of the identity on $H^\pm(U)$.

3.8 Remark. Note that in the case of a spherical domain U, no monomial p_k belongs to $H^\pm(U)$. The situation changes drastically if U has two components V and W with $0 \in V_+$ and $\infty \in W_-$. In this case, $p_{k,V} \in H^+(V)$ for $k \in \mathbb{N}_0$ and and $p_{k,W} \in H^-(W)$ for $k \in -\mathbb{N}$. If Φ has a zero at some integer, then $\Phi(\vartheta)^\pm : H^\pm(U) \to H^\pm(U)$ is no longer injective (according to Remark 4.1 below, at least one of the operators $T_{\varphi,V}^+$ and $T_{\varphi,W}^-$ is not injective).

4 Results for more general Ω

So far we have only considered the case that Ω is of the form Ω_K for some $K \in \mathcal{K}$, which implies that the Mellin transform of φ exists. Our aim now is to obtain a result for more general spherically open Ω, for instance of the form $\Omega = \Omega_{K_1} \cap \Omega_{K_2}$ with $K_1, K_2 \in \mathcal{K}$. It turns out that in some sense non-existence

of the Mellin transform may be compensated by imposing conditions on the number of non-vanishing coefficients φ_k.
We define

$$N_{\varphi^+} := \{k \in \mathbb{N}_0 : \varphi_k = 0\}, \quad N_{\varphi^-} := \{k \in \mathbb{N} : \varphi_{-k} = 0\}$$

and $N_\varphi := N_{\varphi^+} \cup (-N_{\varphi^-})$. From (7) it follows immediately that

$$\operatorname{span}\{p_k : k \in N_\varphi\} \subset \ker(T_\varphi)$$

and

$$\operatorname{span}\{p_k : k \in \mathbb{Z} \setminus N_\varphi\} \subset \operatorname{im}(T_\varphi)$$

(with $\operatorname{span} \emptyset = \{0\}$). In particular, we see that $N_\varphi = \emptyset$ is necessary for the injectivity of T_φ.

4.1 Remark. To obtain a description of the kernel for the one-sided cases T_φ^+ and T_φ^-, we put

$$H_N(U) := \{f \in H^+(U) : f^+(z) = \sum_{\nu \in N} f_\nu z^\nu\}$$

for $N \subset \mathbb{N}_0$ and

$$H_{-N}(U) := \{f \in H^-(U) : f^-(z) = \sum_{\nu \in N} -f_{-\nu} z^{-\nu}\}$$

for $N \subset \mathbb{N}$. Then the expansions (8) and (9), respectively, show that

- $\operatorname{span}\{p_k : k \in N_{\varphi^+}\} \subset \ker(T_\varphi^+) \subset H_{N_{\varphi^+}}(U)$ and if $\Omega * U$ is connected, then $\ker(T_\varphi^+) = H_{N_{\varphi^+}}(U)$.
- $\operatorname{span}\{p_k : k \in -N_{\varphi^-}\} \subset \ker(T_\varphi^-) \subset H_{-N_{\varphi^-}}(U)$ and if $\Omega * U$ is connected, then $\ker(T_\varphi^-) = H_{-N_{\varphi^-}}(U)$.

As a consequence of the first parts we obtain in the case that U is a domain:

- T_φ^+ is injective if and only if $N_{\varphi^+} = \emptyset$.
- T_φ^- injective if and only if $N_{\varphi^-} = \emptyset$.

For $X \subset [0, \infty)$ without finite accumulation point let $n(r) = n_X(r)$ be the number of $x \in X$ with $x \le r$, where $r > 0$. According to [23, p. 559] (see also [13, p. 178]), the limit

$$d^*(X) := \lim_{\xi \to 1^-} \limsup_{r \to \infty} \frac{n(r) - n(r\xi)}{r - r\xi}$$

exists and is called the *maximal density* of X. If the *density* $d(X)$ of X, i.e. $d(X) := \lim_{r \to \infty} n(r)/r$ exists, then $d(X) = d^*(X)$. Moreover, $d^*(X) = 0$ implies the existence of $d(X)$ (and $d(X) = 0$).

4.2 Theorem. *Let U be a spherical domain and suppose that $B_\delta^* * W$ is a domain for some spherically open set $W \subset U$ and some $0 \leq \delta < 1$.*
1. If $N \subset \mathbb{N}_0$ satisfies $d^*(N) \leq \delta$, then $H_N(U) \cap H^{\pm}(U) = \{0\}$.
2. If $d^*(N_{\varphi^+}) \leq \delta$, then $T_\varphi^{\pm}$ is injective.

Proof.
1. We may assume that $0 \notin N$. According to [23, pp. 562] (see also [13, p. 178]), there exists a set $X \subset [0, \infty) \setminus N$ with $d(N \cup X) = \delta$. Since X is countable, there exists a number $\sigma \in (0, 1)$ such that the set $X' := X + \sigma$ does not intersect the non-negative integers. Then $\{\alpha_n : \ n \in \mathbb{N}\} := N \cup X'$ is still a superset of N with $d(N \cup X') = \delta$ and $(N \cup X') \cap \mathbb{N}_0 = N$. We set

$$\Psi(\alpha) := \prod_{n=1}^{\infty} \left(1 - \frac{\alpha^2}{\alpha_n^2}\right) \qquad (\alpha \in \mathbb{C}).$$

Then $\Psi \in \mathrm{Exp}(K_\delta)$ (see Example 2.3.2) with zeros exactly at the points $\pm\alpha_n$. Hence, $\psi := M^{-1}\Psi \in H^{\pm}(B_\delta^*)$ with $N_{\psi^+} = N$.
By assumption, there exists a spherically open set $W \subset U$ such that $B_\delta^* * W$ is connected. With Remark 4.1 and Theorem 3.5 we conclude

$$H_N(W) \cap H^{\pm}(W) = \ker(T_\psi^{+}) \cap H^{\pm}(W) \subset \ker(T_\psi^{\pm}) = \{0\}.$$

Because U is a domain, we get $H_N(U) \cap H^{\pm}(U) = \{0\}$.
2. With Remark 4.1 and 1. we obtain

$$\ker(T_\varphi^{\pm}) \subset H_{N_{\varphi^+}}(U) \cap H^{\pm}(U) = \{0\},$$

□

4.3 Remark. 1. If we have $\delta = 0$ in Theorem 4.2, then we can choose $U = W$ and thus the assertions hold for all spherical domains U. The first assertion can be interpreted in the following way:
Whenever a power series about zero whose non-vanishing coefficients have density zero can be analytically continued up to infinity, then the power series must vanish. Interpreted this way, the assertion is a special case of the Fabry gap theorem (see e.g. [10, Section 11.7], [28, Section 6.4]).
2. If U contains a keyhole domain W of the form

$$W_{r,R}(\eta) = \{0 < |z| < r\} \cup \{|z| > R\} \cup \{z : |\arg(z)| < \pi\eta\}$$

for some $\eta > \delta$ (and some $0 < r < R < \infty$), then $B_\delta^* * W_{r,R}(\eta) = W_{r,R}(\eta - \delta)$ is again a keyhole domain of the above form and thus in particular a spherical domain. The first assertion of Theorem 4.2 shows that whenever a power series about zero whose non-vanishing coefficients have maximal density δ can be analytically continued into a keyhole domain of the above form with $\eta > \delta$, then the power series must vanish. Similarly as in 1. this can now be seen as a special case of the Pólya gap theorem (see [23], [14, p. 3]).
3. According to symmetry (replace φ by $\varphi(1/z)/z$) the conditions on N_{φ^+} can be replaced by the same conditions on N_{φ^-}.

4.4 Corollary. Let $U \subset \mathbb{C}_*$ be a spherical domain. Each of the following conditions is sufficient for injectivity of $T_\varphi^\pm$:

1. $d(N_{\varphi^+}) = 0$
2. $d^*(N_{\varphi^+}) = \delta < 1$ and U contains a keyhole domain $W_{r,R}(\eta)$ for some $\eta > \delta$ (and some $0 < r < R < \infty$).

As a dual version of Theorem 4.2 we get

4.5 Theorem. *Let Ω be spherically open and let $U \subset \mathbb{C}_*$ be open and so that $\Omega * U$ has simply connected components. If $\delta := d^*(N_{\varphi^+}) < 1$ then T_φ has dense range if every open set $V \supset \Omega^* U^*$ contains a spherically open W such that $B_\delta^* * W$ is connected.*

Proof. We show that the transposed operator is injective.
Let $[(g, V)]_{(\Omega^* U^*)_\pm} \in H((\Omega^* U^*)^\pm)$ with $[(g * \varphi, V * \Omega)]_{(U^*)_\pm} = [0]_{(U^*)_\pm}$. Then $g \in H_{N_{\varphi^+}}(V_\pm)$. Since $\Omega^* U^*$ is connected we can choose V to be connected too. Then Theorem 4.2 (with V instead of U) shows that $g = 0$ on V and thus T'_φ is injective. □

4.6 Corollary. Let Ω be spherically open and let U be open and so that $\Omega * U$ has simply connected components. Each of the following conditions is sufficient for T_φ to have dense range

1. $d(N_{\varphi^+}) = 0$.
2. $d^*(N_\varphi^+) = \delta < 1$ and $\Omega * U$ omits a closed cone $\{z : |\arg(z)| \geq \pi(1 - \delta)\}$ of opening $2\pi\delta$.

4.7 Example. We consider the spherical domain $\Omega := \mathbb{C}_* \setminus \{-2, 1\}$ (not being of the form Ω_K for some $K \in \mathcal{K}$) and the function $\varphi \in H(\Omega)$ given by

$$\varphi(z) := \frac{1}{1 - z} + \frac{1}{2 + z} \qquad (z \in \mathbb{C}_* \setminus \{-2, 1\}).$$

Here we have $N_{\varphi^+} = \emptyset$. Corollary 4.4 yields that $T_\varphi^\pm : H(\mathbb{C}_* \setminus \{1\}) \to H(\Omega)$ is injective and Corollary 4.6 shows that $T_\varphi : H(\mathbb{C}_-) \to H(\mathbb{C} \setminus \mathbb{R})$ has dense range.

4.8 Remark. We briefly consider the one-sided cases as in Remark 4.1. From (7) and (8) it follows immediately that (for arbitrary spherically open Ω and arbitrary U so that $U \cup \{0\}$ is open) the condition $N_{\varphi^+} = \emptyset$ is necessary for T_φ^+ to have dense range. Similarly, $N_{\varphi^-} = \emptyset$ is a necessary condition for T_φ^- to have dense range. As an application of Runge's approximation theorem we get

- If $(\Omega * U)_+$ has simply connected components, then T_φ^+ has dense range if and only if $N_{\varphi^+} = \emptyset$.
- If $(\Omega * U)_-$ has simply connected components, then T_φ^- has dense range if and only if $N_{\varphi^-} = \emptyset$.

The results show that the questions concerning injectivity and denseness of the range are easy to answer in the one-sided cases T_φ^+ and T_φ^- (at least under certain natural conditions on U and $\Omega * U$, respectively).

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References

[1] Berenstein C.A., Gay, R. *Complex Variables. An Introduction*, Graduate Texts in Mathematics, 125, Springer, New York, 1991.

[2] Berenstein, C.A., Gay, R. *Complex Analysis and Special Topics in Harmonic Analysis*, Spriger, New York, 1995.

[3] Boas, R.P., *Entire Functions*, Academic Press, New York, 1954.

[4] Brück, R., Müller, J., Invertible elements in a convolution algebra of holomorphic functions, *Math. Ann.* **294** (1992), 421-438.

[5] Domański, P., Langenbruch, M., Representation of multipliers on spaces of real analytic functions, *Analysis (Munich)* **32** (2012), 137-162.

[6] Domański, P., Langenbruch, M., Algebra of multipliers on the space of real analytic functions of one variable, *Studia Math.*, **212** (2012), 155-171.

[7] Domański, P., Langenbruch, M., Hadamard multipliers on spaces of real analytic functions, *Adv. Math.*, **240** (2013), 575-612.

[8] Frerick, L., Coefficient multipliers with closed range, *Note Mat.*, **17** (1997), 61-70.

[9] Große-Erdmann, K.-G., On the Borel-Okada Theorem and the Hadamard Multiplication Theorem, *Complex Var. Theory Appl.*, **22** (1993), 101 - 112.

[10] Hille, E., *Analytic Function Theory*, Volume II, Chelsea Publishing Company, New York, 1962.

[11] Ishimura, R., Existence locale de solutions holomorphes pour les équations différentielles d'ordre infini, *Ann. Inst. Fourier*, **35** (1985), 49-57.

[12] Ishimura, R., Sur les équations différentielles d'ordre infini d'Euler, *Mem. Fac. Sci. Kyushu Univ. Ser. A*, **44** (1990), 1-10.

[13] Koosis, P. *The Logarithmic Integral, I*, Cambridge University Press, Cambridge, 1988.

[14] Koosis, P. *The Logarithmic Integral, II*, Cambridge University Press, Cambridge, 1992.

[15] Korobeĭnik, J. F.: Existence of an analytic solution of a differential equation of infinite order and the character of its domain of analyticity (Russian), *Mat. Sb.* (N. S.), **80** (1969), 52-76, English translation: *Math. USSR-Sb.* **9** (1969), 53-71.

[16] Krivosheev, A.S., Napalkov, V.V., Complex analysis and convolution operators, *Uspekhi Mat. Nauk*, **47** (1992), 3-58 (Russian); English transl. *Russian Math. Surveys*, **47** (1992), 1-56.

[17] Levin, B.Y., *Nullstellenverteilung ganzer Funktionen*, Akademie Verlag, Berlin, 1962.

[18] Lorson, T., *Hadamard Convolution Operators on Spaces of Holomorphic Functions*, Dissertation, University of Trier, URL: http://ubt.opus.hbz-nrw.de/volltexte/2014/840/

[19] Meise R., Vogt, D., *Introduction to Functional Analysis*, Oxford University Press, Oxford, 1997.

[20] Müller, J., The Hadamard Multiplication Theorem and Applications in Summability Theory, *Complex Var. Theory Appl.*, **18** (1992), 155-166.

[21] Müller, J., Coefficient multipliers from $H(G_1)$ into $H(G_2)$, *Arch. Math.*, **61** (1993), 75-81.

[22] Müller, J., Pohlen, T., The Hadamard product on open sets in the extended plane, *Complex Anal. Oper. Theory*, **6** (2012), 257-274.

[23] Pólya, G., Untersuchungen über Lücken und Singularitäten von Potenzreihen, I., *Math. Z*, **29** (1929), 549-640.

[24] Render, H., Hadamard's multiplication theorem – recent developments, *Colloquium Math.*, **74** (1997), 79-92.

[25] Render, H., Sauer, A., Algebras of holomorphic functions with Hadamard muliplication, *Studia Math.*, **118** (1996), 77-100.

[26] Render, H., Sauer, A., Invariance properties of homomorphisms of algebras of holomorphic functions with the Hadamard product, *Studia Math.*, **121** (1996), 53-65.

[27] Rudin, W., *Real and Complex Analysis* (3rd ed.), McGraw-Hill, New York, 1987.

[28] Segal, S.L., *Nine Introductions in Complex Analysis* (revised ed.), Elsevier, Amsterdam, 2008.