UNIVERSALITY VS. NON-NORMALITY OF FAMILIES OF MEROMORPHIC FUNCTIONS

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Abstract. For a family $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ of meromorphic functions on an open set $\Omega \subset \mathbb{C}$, we will establish several connections between the property that $\mathcal{F}$ is a universal family, i.e. that restrictions of $\mathcal{F}$ to suitable subsets of Ω are dense families in the corresponding function spaces, and the property that $\mathcal{F}$ is a non-normal family.

1. Introduction

Consider $f : \mathbb{C} \to \mathbb{C}$ to be a non-affine entire function. When studying the dynamics of the iterates of f, concepts and notions from the theory of non-normal families and from topological dynamics are usually intertwined (see e.g. [1, 16, 20]): The Julia set $J = J(f)$ is defined as the set of all $z \in \mathbb{C}$ such that the family of iterates $\{f^{\circ n} : n \in \mathbb{N}\}$ is not normal at z. Montel's theorem and the complete invariance of J imply that the dynamical system (J, f) is topologically transitive. Combined with Birkhoff's transitivity theorem, this shows that for a comeagre set of points $z \in J$ the orbits $\{f^{\circ n}(z) : n \in \mathbb{N}\}$ are dense in J. An application of Zalcman's lemma then results in the basic property that the repelling periodic points are dense in J. In turn, this implies that in the case of a polynomial f the dynamical system is topologically exact, that is, for each non-empty relatively open set $V \subset J$ there is $n \in \mathbb{N}$ with $f^n(V) = J$.

In the past years, many notions from topological dynamics have been extended to the more general framework of universality of families of functions (see e.g. [11, 2, 12]). This leads to the idea to systematically study, for families of holomorphic or meromorphic functions, the relationship between the property of being a non-normal family and the property of being universal. In this paper, some steps in this direction will be done.

We start with some auxiliary results on products of families of continuous functions (Section 2). In Section 3, we will focus on universality properties of sequences of meromorphic functions that are non-normal on open sets. Subsequently, Section 4 provides assumptions under which sequences of meromorphic functions that are non-normal on more general sets have universality properties.

We recall (or introduce) some relevant notions in a general framework. For topological spaces X, Y, we denote by $C(X, Y)$ the space of all continuous functions from X to Y, which shall always be endowed with the compact-open topology. For $B \subset Y$ we say that an infinite family $\mathcal{F} \subset C(X, Y)$ is *topologically transitive with respect to* B, if for all non-empty open sets $U \subset X$ and $V \subset Y$ with $B \cap V \neq \varnothing$ there exists some $f \in \mathcal{F}$ with $f(U) \cap V \neq \varnothing$. Note that $\mathcal{F}$ is topologically transitive

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with respect to B if and only if it is topologically transitive with respect to the closure $\overline{B}$ of B. So we consider B to be closed in the sequel. If $f(U) \cap V \neq \varnothing$ holds for cofinitely many $f \in \mathcal{F}$, then the family is called *topologically mixing with respect to* B. Moreover, we say that the family $\mathcal{F}$ is *hereditarily transitive with respect to* B, if an infinite subfamily $\mathcal{F}_0$ exists with the property that each infinite subset of $\mathcal{F}_0$ is topologically transitive with respect to B. In the case $B = Y$, the suffix *with respect to* B is suppressed.

A sequence (f_n) of continuous functions $f_n : X \to Y$ is called *universal* if there exists an element $x \in X$ such that the set $\{f_n(x) : n \in \mathbb{N}\}$ is dense in Y. Elements with this property are called *universal for* (f_n). We say that a property is fulfilled for *generically many* elements of a complete metric space if it is fulfilled on a comeagre subset of the space. If X is a complete metric space and Y is a separable metric space, the Universality Criterion (see e.g. [12, Theorem 1.57]) states that a dense set of universal elements for (f_n) exists if and only if $\{f_n : n \in \mathbb{N}\}$ is topologically transitive. In this case, generically many elements are universal for (f_n).

Finally, in the case of metric spaces X, Y, a family $\mathcal{F} \subset C(X, Y)$ is called *normal* if each sequence in $\mathcal{F}$ has a subsequence which converges uniformly on each compact subset of X. Moreover, $\mathcal{F}$ is called *normal at a point* $x \in X$ if there exists a neighbourhood W of x such that the family $\mathcal{F}|_W := \{f|_W : f \in \mathcal{F}\}$ is normal in $C(W, Y)$. If X is locally compact and has a compact exhaustion (i.e. there exists a sequence (K_n) of compact subsets of X with $X = \bigcup_{n\in\mathbb{N}} K_n$ and $K_n \subset K_{n+1}^\circ$ for all $n \in \mathbb{N}$), it can be shown that normality is a local property, i.e. a family $\mathcal{F} \subset C(X, Y)$ is normal if and only if it is normal at each point $x \in X$ (cf. [18], Theorem 2.1.2, in case that X is a domain in $\mathbb{C}$).

2. Products of families

For $N \in \mathbb{N}$, sets $X_1, \dots, X_N, Y_1, \dots, Y_N$, and functions $f_j : X_j \to Y_j$ $(j = 1, \dots, N)$, we denote by

$$f_1 \times \cdots \times f_N : \prod_{j=1}^N X_j \to \prod_{j=1}^N Y_j, \ \ (x_1, \dots, x_N) \mapsto (f_1(x_1), \dots, f_N(x_N))$$

the N-fold product of $f_1, \dots, f_N$. In case of $f_j = f$ for all $j \in \{1, \dots, N\}$, we write $f^{\times N} := f_1 \times \cdots \times f_N$. For topological spaces X, the product X^N shall always be equipped with the product topology.

Proposition 2.1. *Let X, Y be metric spaces with X locally compact, let $\mathcal{F} \subset C(X, Y)$ and for $N \in \mathbb{N}$ consider the corresponding N-fold families*

$$\mathcal{F}^{\times N} := \left\{f^{\times N} : f \in \mathcal{F}\right\} \subset C(X^N, Y^N).$$

(1) *If $x = (x_1, \dots, x_N) \in X^N$ then $\mathcal{F}^{\times N}$ is normal at x if and only if $\mathcal{F}$ is normal at $x_1, \dots, x_N$.*
(2) *If X has a compact exhaustion then $\mathcal{F}^{\times N}$ is normal if and only if $\mathcal{F}$ is normal.*

Proof. 1. If $\mathcal{F}^{\times N}$ is normal at $x = (x_1, \dots, x_N)$ then there exist open neighbourhoods U_j of x_j $(j = 1, \dots, N)$ such that $\mathcal{F}^{\times N}|_{U_1\times\cdots\times U_N}$ is a normal family. But this already implies that $\mathcal{F}$ is normal at each point x_j. Indeed, given a sequence (g_k) in $\mathcal{F}|_{U_j}$, there exists a sequence (f_k) in $\mathcal{F}$ with $g_k = f_k|_{U_j}$ for all $k \in \mathbb{N}$.

As $(f_k|_{U_1} \times \cdots \times f_k|_{U_N})$ is a sequence in $\mathcal{F}^{\times N}|_{U_1\times\cdots\times U_N}$, the normality of the family $\mathcal{F}^{\times N}|_{U_1\times\cdots\times U_N}$ yields the existence of a strictly increasing sequence (k_l) in $\mathbb{N}$ such that $(f_{k_l}|_{U_1} \times \cdots \times f_{k_l}|_{U_N})$ converges uniformly on each compact subset of $U_1\times\cdots\times U_N$. Clearly, this implies that $(g_{k_l}) = (f_{k_l}|_{U_j})$ converges locally uniformly on U_j.

If, conversely, $\mathcal{F}$ is normal at $x_1, \ldots, x_N$ then for $j = 1, \ldots, N$ there are compact neighbourhoods K_j of x_j such that $\mathcal{F}|_{K_j}$ is normal. If $(f_n^{\times N})$ is a sequence in $\mathcal{F}^{\times N}$, then by a standard sub-sub-sequence argument it is seen that a subsequence $(f_{n_k}^{\times N})$ converges uniformly on $K_1 \times \cdots \times K_N$. This shows that $\mathcal{F}^{\times N}$ is normal at x.

2. The second statement follows directly from 1. and the fact that normality is a local property under the corresponding assumptions (cf. the consideration at the end of Section 1). □

Remark 2.2. Considering for $N \in \mathbb{N}$ the larger families

$$\mathcal{F}_N := \{f_1 \times \cdots \times f_N : f_j \in \mathcal{F} \text{ for all } j = 1, \ldots, N\} \subset C(X^N, Y^N)$$

the proof of Proposition 2.1 shows that, under the corresponding assumptions on X, normality of $\mathcal{F}$ is also equivalent to normality of $\mathcal{F}_N$.

In view of Proposition 2.1, in general there is a basic difference between topological transitivity and non-normality of N-fold products of sequences of continuous functions. Indeed, there exist examples of topologically transitive dynamical systems (X, T) (that is, the family of iterates $\{T^{\circ n} : n \in \mathbb{N}\}$ is topologically transitive) for which (X, T) is not topologically weak-mixing ((X, T) is called *topologically weak-mixing*, if $(X^2, T^{\times 2})$ is topologically transitive). For instance, given $\alpha \in \mathbb{R} \setminus \pi\mathbb{Q}$, the circle rotation $(\mathbb{T}, f)$ defined by $f(z) := e^{i\alpha}z$, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$, is topologically transitive but not topologically weak-mixing (cf. [12, Example 1.43]). By the famous De la Rosa-Read theorem, there even exist continuous linear operators T on Banach spaces X such that (X, T) is topologically transitive but not weak-mixing (cf. [7], see also [12, Theorem 2.43]). However, due to Furstenberg's theorem (see, e.g. [12, Theorem 1.51]), $(X^N, T^{\times N})$ is topologically transitive for all $N \in \mathbb{N}$ if (X, T) is topologically weak-mixing. In the sequel, we say that a family $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ of continuous functions $f_n : X \to Y$ is *topologically weak-mixing with respect to* $B \subset Y$ if $\mathcal{F}^{\times N}$ is topologically transitive with respect to B^N for all $N \in \mathbb{N}$.

The Bès-Peris Theorem (see e.g. [12, Theorem 3.15]) shows that a linear system (X, T) is hereditarily transitive if and only if it is topologically weak-mixing. For arbitrary families we have the following version of the Bès-Peris Theorem:

Proposition 2.3. *Let X, Y be metric spaces and $B \subset Y$ be a closed set. Then $\mathcal{F} \subset C(X, Y)$ is hereditarily transitive with respect to B if and only if $\mathcal{F}$ is topologically weak-mixing with respect to B.*

Proof. $\Rightarrow$: Let $\mathcal{F}_0 = \{f_n : n \in \mathbb{N}\} \subset \mathcal{F}$ be so that each infinite subset is transitive. Let $N \in \mathbb{N}$, $U \subset X^N$ be open and non-empty, and $V \subset Y^N$ open with $B^N \cap V$ non-empty. Then there exist non-empty and open sets $U_1, \ldots, U_N \subset X$ with $U_1 \times \cdots \times U_N \subset U$ and $V_1, \ldots, V_N \subset Y$ with $V_1 \times \cdots \times V_N \subset V$ and $B \cap V_1, \ldots, B \cap V_N$ non-empty. According to the assumption, $\{f_n : n > m\}$ is topologically transitive with respect to B, for all $m \in \mathbb{N}$. Inductively, we can find a strictly increasing sequence (n_k) in $\mathbb{N}$ with $f_{n_k}(U_1) \cap V_1 \neq \varnothing$ for all $k \in \mathbb{N}$. By assumption, the family $\{f_{n_k} : k \in \mathbb{N}\}$ is topologically transitive with respect to B. Thus, the same argument as above yields the existence of a subsequence $(n_k^{(2)})$ of $(n_k^{(1)}) := (n_k)$ with $f_{n_k^{(2)}}(U_2) \cap V_2 \neq \varnothing$ for all $k \in \mathbb{N}$. Proceeding in the same way, for any $2 \leq j \leq N$ we find subsequences $(n_k^{(j)})$ of $(n_k^{(j-1)})$ with $f_{n_k^{(j)}}(U_j) \cap V_j \neq \varnothing$ for all $k \in \mathbb{N}$. In particular, for $n := n_1^{(N)}$, we obtain that

$$\varnothing \neq (f_n(U_1) \times \cdots \times f_n(U_N)) \cap (V_1 \times \cdots \times V_N) \subset f_n^{\times N}(U) \cap V,$$

i.e. $\mathcal{F}^{\times N}$ is topologically transitive with respect to B^N.

$\Leftarrow$: The proof follows along the same lines as the proof of the corresponding part of the Bès-Peris Theorem (see e.g. [12, pp. 76]). $\square$

3. Non-normality on large sets

We denote by $\mathbb{C}_\infty$ the extended complex plane, which shall be endowed with the spherical metric. We recall that, for arbitrary $E \subset \mathbb{C}$, the space $C(E, \mathbb{C}_\infty)$ of continuous functions $f : E \to \mathbb{C}_\infty$ is endowed with the compact-open topology. Thus, convergence in $C(E, \mathbb{C}_\infty)$ means uniform convergence with respect to the spherical metric on all compact subsets of E. For $E' \supset E$ and a family $\mathcal{F} \subset C(E', \mathbb{C}_\infty)$ we write $\omega(\mathcal{F}, E)$ for the sequential closure of $\mathcal{F}|_E$ in $C(E, \mathbb{C}_\infty)$, that is, the set of all $h \in C(E, \mathbb{C}_\infty)$ which are uniform limit on arbitrary compact sets in E of some sequence in $\mathcal{F}$. For $z \in \mathbb{C}$ we write $\omega(\mathcal{F}, z) := \omega(\mathcal{F}, \{z\})$.

Let now $\Omega \subset \mathbb{C}$ be an open set. We write $H(\Omega)$ for the space of holomorphic functions on Ω and $M(\Omega)$ for the space of meromorphic functions on Ω. We allow meromorphic functions to be locally constant ∞, which makes $M(\Omega)$ a closed subspace of $C(\Omega, \mathbb{C}_\infty)$. Moreover, $H_\infty(\Omega)$ denotes the closure of $H(\Omega)$ in $M(\Omega)$. Then $f \in H_\infty(\Omega)$ if and only if f is holomorphic or constant ∞ in each component of Ω. According to the Arzelà-Ascoli theorem, a family $\mathcal{F} \subset M(\Omega)$ is normal if and only if it is (spherically) equicontinuous.

Remark 3.1. Let $\mathcal{F} \subset M(\Omega)$ and $A \subset \Omega$ closed in $\mathbb{C}$.

1. Let $N \in \mathbb{N}$. According to the Universality Criterion, $\mathcal{F}^{\times N}|_{A^N}$ is topologically transitive if and only if $\omega(\mathcal{F}, E) = (\mathbb{C}_\infty)^E$ for generically many $(z_1, \ldots, z_N) \in A^N$ and $E := \{z_1, \ldots, z_N\}$. It is easily seen that the following extension holds: If $B \subset \mathbb{C}_\infty$ is closed then the family $\mathcal{F}^{\times N}|_{A^N}$ is topologically transitive with respect to B^N if and only if $\omega(\mathcal{F}, E) \supset B^E$ for generically many $(z_1, \ldots, z_N) \in A^N$ and $E := \{z_1, \ldots, z_N\}$ (cf. [10, Satz 1.2.2]).

2. If X is a metric space, $\mathcal{K}(X)$ shall denote the set of all compact non-empty subsets of X, and we endow $\mathcal{K}(X)$ with the Hausdorff metric. It is known that $\mathcal{K}(X)$ is complete whenever X is complete (see e.g. [8, Section 2.4]). Corollary 1.2 in [3] shows that $\omega(\mathcal{F}, K) = C(K, \mathbb{C}_\infty)$ for generically many $K \in \mathcal{K}(A)$ if $\mathcal{F}|_A$ is topologically weak-mixing (note that $C(K, \mathbb{C})$ is dense in $C(K, \mathbb{C}_\infty)$). In a similar way, the proofs of Theorem 1.1 and Corollary 1.2 in [3] show that $\omega(\mathcal{F}, K) \supset C(K, B)$ for generically many $K \in \mathcal{K}(A)$ if $\mathcal{F}|_A$ is topologically weak-mixing with respect to B.

It is easily seen that topologically transitive families of holomorphic functions can be normal:

Example 3.2. Let $\Omega \subset \mathbb{C}$ be an open set and let $\mathcal{F} \subset H_\infty(\Omega)$ be the family of all functions which are locally constant on Ω. Since $\mathcal{F}$ is obviously locally normal at all points, the family is normal on Ω. On the other hand, for each set $E \subset \Omega$

we have $\omega(\mathcal{F},E) = \mathcal{F}|_E$. In particular, $\omega(\mathcal{F},z) = \mathbb{C}_\infty$ for all $z \in \Omega$ and thus, by Remark 3.1.1 (with $N = 1$), the family $\mathcal{F}$ is topologically transitive.

Note that in the preceding example $\omega(\mathcal{F},E)$ consists solely of locally constant functions. More generally, the following extension of Vitali's theorem holds:

Proposition 3.3. *If $U \subset \mathbb{C}$ is open and $\mathcal{F} \subset M(U)$ is a normal family, then for all $E \subset U$ the restriction mapping $\omega(\mathcal{F},U) \ni h \mapsto h|_E \in \omega(\mathcal{F},E)$ is surjective. If U is a domain and E has an accumulation point in U, then the mapping is also injective.*

Proof. Let $g \in \omega(\mathcal{F},E)$ and (f_n) a sequence in $\mathcal{F}$ converging to g in $C(E,\mathbb{C}_\infty)$. Since a subsequence of (f_n) converges locally uniformly on U there exists a function $h \in M(U)$ such that $h|_E = g$. If U is a domain and E has an accumulation point, the identity theorem shows that the restriction mapping is injective. $\square$

Proposition 3.4. *Let $\Omega \subset \mathbb{C}$ be an open set and let $B \subset \mathbb{C}_\infty$ contain at least two points a, b. If $\mathcal{F} \subset M(\Omega)$ is so that $\mathcal{F}^{\times 2}$ is topologically transitive with respect to B^2, then $\mathcal{F}$ is not normal at any point of Ω.*

Proof. Suppose $\mathcal{F}$ is normal at some point in $a \in \Omega$. Then $\mathcal{F}$ is equicontinuous at a. Let $A_k := \{z : |z-a| \le 1/k\}$, where k is so large that $A_k \subset \Omega$. Since $\mathcal{F}^{\times 2}$ is topologically transitive with respect to B^2, the same holds for $\mathcal{F}|_{A_k}$. According to Remark 3.1.1, for each k there is a two-point set $E_k = \{z_k, w_k\} \subset A_k$ with $\omega(\mathcal{F},E_k) \supset B^{E_k}$. Hence, for each k there is some $f_k \in \mathcal{F}$ with

$$\chi(f_k(z_k), f_k(w_k)) \ge \chi(b,c)/2.$$

But this contradicts the equicontinuity of $\mathcal{F}$ at a. $\square$

In the converse direction, we now start with non-normal families $\mathcal{F} \subset M(\Omega)$. We say that $\mathcal{F}$ is *hereditarily non-normal on* $E \subset \Omega$, if for some infinite subfamily $\mathcal{F}_0$ of $\mathcal{F}$ each infinite subfamily of $\mathcal{F}_0$ is not normal at any point of E. If $g^\#$ denotes the spherical derivative of $g \in M(\Omega)$, then, according to Marty's theorem (see e.g. [9, p. 318]), $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ is hereditarily non-normal on E if and only if for some subsequence (n_k) of (n)

$$\sup_{z \in U} (f_{n_k})^\#(z) \to \infty \quad (k \to \infty) \tag{3.1}$$

holds for arbitrary open sets U that meet E.

It can be easily shown that non-normality of a family at all points of a set E does not always imply hereditary non-normality on E:

Example 3.5. The family $\{(z-a)^k : k \in \mathbb{N}\}$ is not normal at any point of the circle $\mathbb{T} + a$ with centre a. If we define $f_{2k}(z) := (z-2)^k$ and $f_{2k-1}(z) := (z+2)^k$, then $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ is not normal at any point of $(\mathbb{T}+2) \cup (\mathbb{T}-2)$. On the other hand, let E be any set that intersects both $\mathbb{T}+2$ and $\mathbb{T}-2$. If $\mathcal{F}_0 = \{f_{n_j} : j \in \mathbb{N}\}$ is a subfamiliy of $\mathcal{F}$, then infinitely many of the n_j are even or infinitely many are odd. Thus $\mathcal{F}_0$ has an infinite subfamily which is normal at some point in E, and so $\mathcal{F}$ is not hereditarily non-normal on E. In particular, this holds for $E = (\mathbb{T}+2) \cup (\mathbb{T}-2)$.

Proposition 3.6. *Let $\Omega \subset \mathbb{C}$ be an open set, $\mathcal{F} \subset M(\Omega)$ and $A \subset \Omega$ with dense interior A°.*

1. *If $\mathcal{F}$ is not normal at any point of A, then $\mathcal{F}|_A$ is topologically transitive.*
2. *If $\mathcal{F}$ is hereditarily non-normal on A, then $\mathcal{F}|_A$ is topologically weak-mixing.*

Proof. 1. Let $U \subset A^\circ$ and $V \subset \mathbb{C}_\infty$ be open and non-empty. Since $\mathcal{F}|_U$ is not normal, Montel's theorem (see e.g. [18]) shows that $\mathbb{C}_\infty \setminus \bigcup_{f\in\mathcal{F}} f(U)$ contains at most two points, so $V \cap \bigcup_{f\in\mathcal{F}} f(U)$ is non-empty. In particular, $\mathcal{F}|_{A^\circ}$ is topologically transitive. Since A° is dense in A, also $\mathcal{F}|_A$ is topologically transitive.

2. According to the assumption, there exists an infinite subfamily $\mathcal{F}_0$ of $\mathcal{F}$ such that each infinite subset of $\mathcal{F}_0$ is not normal at any point of A, and so each infinite subset of $\mathcal{F}_0|_A$ is topologically transitive by part 1. Now, the assertion follows from Proposition 2.3 which shows that $\mathcal{F}_0|_A$ is topologically weak-mixing. □

Combining the preceding propositions, we obtain the following assertion.

Theorem 3.7. *For $\Omega \subset \mathbb{C}$ open and $\mathcal{F} \subset M(\Omega)$ the following are equivalent:*

(1) *$\mathcal{F}$ is hereditarily non-normal on Ω.*
(2) *$\mathcal{F}$ is hereditarily transitive.*
(3) *$\mathcal{F}$ is topologically weak-mixing.*

Proof. The equivalence of (2) and (3) follows from Proposition 2.3, and that (1) implies (3) is part of Proposition 3.6. Finally, if (3) holds, then also $\mathcal{F}^{\times 2}$ is topologically weak-mixing and thus, according to Proposition 2.3, hereditarily transitive. Proposition 3.4 now shows that (1) holds. □

The next lemma supplies examples of hereditary non-normality. We write M_0 for the set of $f \in M(\mathbb{C})$ which are either a non-affine entire function or a function having exactly one pole in $\mathbb{C}$ that, in addition, is an omitted value. Without loss of generality this value can (and will) be chosen to be 0. We write $D := D(f) := \mathbb{C}$ in the first case and $D := D(f) := \mathbb{C} \setminus \{0\}$ in the second. Then $f(D) \subset D$ and thus $f^{\circ n}$ are defined and holomorphic in D. The Julia set $J = J(f)$ is defined as the set of all $z \in D(f)$ such that $\mathcal{F} := \{f^{\circ n} : n \in \mathbb{N}\}$ is not normal at z. It is well-known that J has either empty interior or equals D. Moreover, J is always a perfect set (that is, J is non-empty, D-closed and each point is an accumulation point) and in the case of polynomials J is compact in $\mathbb{C}$. For these and more results we refer to [5].

Lemma 3.8. *For all functions $f \in M_0$ the family $\{f^{\circ n} : n \in \mathbb{N}\}$ is hereditarily non-normal on J.*

Proof. We choose a countable dense subset $\{w_k : k \in \mathbb{N}\}$ of J. It is known that the repelling periodic points are dense in J (see e.g. [5, Theorem 4]). Hence, there are $z_1 \in J$ and $q_1 \in \mathbb{N}$ with $|z_1 - w_1| < 1$ and $f^{\circ(jq_1)}(z_1) = z_1$ for all j. Since

$$(f^{\circ(jq_1)})'(z_1) = ((f^{\circ q_1})'(z_1))^j =: \lambda^j$$

and

$$(f^{\circ(jq_1)})^\#(z_1) = |(f^{\circ(jq_1)})'(z_1)|/(1 + |z_1|^2) = |\lambda|^j/(1 + |z_1|^2),$$

we can choose $j_1 \in \mathbb{N}$ with $(f^{\circ(jq_1)})^\#(z_1) > 1$ for all $j \geq j_1$ because $|\lambda| > 1$. Since $J(f^{\circ q_1}) = J$ (see [5, Lemma 1]), in a similar way we can find $z_2 \in J$, $q_2 \in q_1\mathbb{N}$ and $j_2 \in \mathbb{N}$ with $|z_2 - w_2| < 1/2$, $j_2 \geq j_1$ and

$$(f^{\circ(jq_2)})^\#(z_2) > 2$$

for all $j \geq j_2$. Inductively, we obtain a sequence (z_k) in J with $|z_k - w_k| < 1/k$, a sequence $(q_k)_k$ with $q_k \in q_{k-1}\mathbb{N}$ (we set $q_0 := 1$) and a sequence (j_k) in $\mathbb{N}$ with $j_k \geq j_{k-1}$ such that

$$(f^{\circ(jq_k)})^\#(z_k) > k$$

for all $j \geq j_k$. By construction, the z_k form a dense set in J. Since J is perfect, for each open set U that meets J there are infinitely many k with $z_k \in U$. Thus, the construction guarantees that (3.1) is satisfied. □

Remark 3.9. If f is a non-affine entire function, then from the denseness of the repelling periodic points in J and the fact that J is the boundary of the escaping set $I(f) = \{z : f^{\circ n}(z) \to \infty\}$ (see e.g. [19, Theorem 4.1]) it follows that indeed each infinite subfamily of $\{f^{\circ n} : n \in \mathbb{N}\}$ is non-normal on J.

Example 3.10. If $f \in M_0$ has the Julia set $J = D$, then Proposition 3.6 and Lemma 3.8 imply that $\{f^{\circ n} : n \in \mathbb{N}\}$ is topologically weak-mixing. According to Remark 3.1.2, $\omega(f, K) = C(K, \mathbb{C}_\infty)$ for generically many $K \in \mathcal{K}(J)$ (cf. also [4, Theorem 1] and [3, Theorem 2.1]).

In Section 4, Lemma 3.8 will be applied to an example of transitivity with respect to a non-necessarily open subset of $\mathbb{C}$.

Remark 3.11. Connections between the Julia set of a complex polynomial and the chaotical dynamics of a certain associated nonlinear operator on sequence spaces can be found in [13] and [17].

4. Non-normality on saturated sets

In this section, we consider sequences (f_n) of functions meromorphic in an open set Ω which are non-normal on arbitrary closed subsets of Ω. Even in case of a strong kind of non-normality of such a family it might be the case that it does not possess any universality properties at all: For $f(z) := \sum_{\nu=0}^{\infty} a_\nu z^\nu$ a power series with radius of convergence 1, let $S_n f$ denote the n-th partial sum of the series. Vitali's Theorem shows that $\{S_n f : n \in \mathbb{N}\}$ is not normal at any point $z \in \mathbb{T}$. If $\sum_{\nu=0}^{\infty} |a_\nu| < \infty$, then the partial sums $(S_n f)$ converge uniformly on $\mathbb{T}$ to f. This implies that $\omega(S_n f, E) = \{f|_E\}$ is a one-point set, for every set $E \subset \mathbb{T}$. Thus, we have no kind of universality behaviour.

In order to state our next result, we recall that a closed set $A \subset \mathbb{C}$ is called *perfect* if A has no isolated points. Perfect sets are locally uncountable, that is, $U \cap A$ is uncountable for all open sets $U \subset \mathbb{C}$ which intersect A. For sets X, Y and a function $f : X \to Y$, a set $A \subset X$ is called *f-saturated* if $A = f^{-1}(f(A))$. It is easily seen that A is f-saturated if and only if $f(A \cap B) = f(A) \cap f(B)$ for all $B \subset X$. If X, Y are topological spaces we say that, for a family $\mathcal{F}$ of functions $f : X \to Y$, the set A is *$\mathcal{F}$-saturated at $x \in X$*, if a neighbourhood U of x exists with $f(A \cap U) = f(A) \cap f(U)$ for all $f \in \mathcal{F}$.

Let $\Omega \subset \mathbb{C}$ be a domain and $(f_n)_{n \in \mathbb{N}}$ a sequence in $M(\Omega)$. We recall that $\liminf f_n(A) = \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} f_k(A)$ for $A \subset \Omega$.

Theorem 4.1. *Let $\Omega \subset \mathbb{C}$ be a domain and $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$ a family in $M(\Omega)$. Suppose that $A \subset \Omega$ is $\mathcal{F}$-saturated at all points $z \in A$ and that $\liminf f_n(A)$ is a locally uncountable set in $\mathbb{C}_\infty$.*

(1) *If $\mathcal{F}$ is not normal at any point of A then $\mathcal{F}|_A$ is topologically transitive with respect to the closure B of $\liminf f_n(A)$. If, in addition, for each relatively open set $W \subset A$ the sequence $(f_n(W))$ is eventually increasing then $\mathcal{F}|_A$ is topologically mixing with respect to B.*
(2) *If $\mathcal{F}$ is hereditarily non-normal on A, then $\mathcal{F}|_A$ is topologically weak-mixing with respect to B.*

Proof. 1. Let $W \subset A$ be relatively open and non-empty. Then an open set $U \subset \Omega$ exists with $U \cap A \subset W$ and so that $f(A) \cap f(U) = f(A \cap U)$ for all $f \in \mathcal{F}$. Moreover, let $V \subset \mathbb{C}_\infty$ be open with $V \cap B \neq \varnothing$. Since $\liminf f_n(A)$ is locally uncountable, we obtain that $V \cap \liminf f_n(A)$ is uncountable. By definition of $\liminf f_n(A)$, there exists some $m \in \mathbb{N}$ such that $V \cap \bigcap_{k \geq m} f_k(A)$ is uncountable. Hence, as $\{f_n|_U : n \geq m\}$ is a non-normal family, Montel's theorem implies that also the set

$$\Big(\bigcup_{n \geq m} f_n(U)\Big) \cap V \cap \bigcap_{k \geq m} f_k(A) = \bigcup_{n \geq m} \Big(f_n(U) \cap V \cap \bigcap_{k \geq m} f_k(A)\Big)$$

is uncountable so that there exists some $n \geq m$ such that

$$M := f_n(U) \cap V \cap \bigcap_{k \geq m} f_k(A)$$

is uncountable, too. According to

$$M \subset f_n(U) \cap f_n(A) \cap V = f_n(U \cap A) \cap V \subset f_n(W) \cap V,$$

it follows that $f_n(W) \cap V$ is uncountable and thus in particular non-empty. This shows that $\mathcal{F}|_A$ is topologically transitive with respect to B. If $f_n(W)$ is eventually increasing, we can choose m above so that, in addition, $(f_n(W))_{n \geq m}$ is increasing. Then $\mathcal{F}$ is topologically mixing with respect to B.

2. Part 1 implies that each infinite subset of $\mathcal{F}_0$ is topologically transitive with respect to B. From Proposition 2.3 it follows that the family $\mathcal{F}_0|_A$ is topologically weak-mixing with respect to B. □

Remark 4.2. Since the $\limsup B_n = \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} B_k$ of a sequence $(B_n)_{n \in \mathbb{N}}$ is the union of the $\liminf(B_{n_k})$ over all subsequences $(B_{n_k})_k$ of (B_n), in part 1 of Theorem 4.1 the set B can be replaced by the closure of $\limsup f_n(A)$ if all infinite subsets of $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ are not normal at any point of A. Indeed, if an open set V meets that closure then it meets $\limsup f_n(A)$, so V meets $\liminf f_{n_k}(A)$ for some (n_k), and then V meets $\overline{\liminf f_{n_k}(A)}$. Therefore part 1 can be applied to this set thanks to our assumption of non-normality.

Examples 4.3.

1. Let (λ_n) be a sequence of real numbers and let

$$f_n(z) := \exp(\lambda_n z)$$

for $n \in \mathbb{N}$ and $z \in \mathbb{C}$. With $\Omega = \mathbb{C}$ and $A = i\mathbb{R}$ we have $f_n(i\mathbb{R}) = \mathbb{T}$ and A is f_n-saturated for all $n \in \mathbb{N}$. Since

$$f_n^{\#}(z) = 2|\lambda_n| / \cosh(\lambda_n \operatorname{Re} z),$$

Marty's theorem implies that for $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ the following are equivalent:

(1) $\mathcal{F}$ is not normal.
(2) $\mathcal{F}$ is not normal at any point of $i\mathbb{R}$.
(3) $\mathcal{F}$ is hereditarily non-normal on $i\mathbb{R}$.
(4) (λ_n) is unbounded.

Recall that a compact set $E \subset i\mathbb{R}$ is said to be a *Kronecker set* if

$$\omega(\{\exp(n\,\cdot) : n \in \mathbb{N}\}, E) = C(E, \mathbb{T}).$$

Theorem 4.1 and Remark 3.1.2 yield that for all unbounded sequences (λ_n) generically many compact subsets E of $i\mathbb{R}$ satisfy $\omega(\mathcal{F}, E) = C(E, \mathbb{T})$ (cf. [14, Section 6.3] or [3, Example 1.3]).

2. As mentioned in the introduction, the iterates $f^{\circ n}|_{\mathbb{T}}$ of an irrational rotation $f(z) = e^{i\alpha}z$ form a topologically transitive family with respect to the $f^{\circ n}$-saturated set $\mathbb{T} = f^{\circ n}(\mathbb{T})$ which is not topologically weak-mixing with respect to $\mathbb{T}$. Note that the family $\{f^{\circ n} : n \in \mathbb{N}\}$ is normal in $\mathbb{C}$.

3. Let f be a function in M_0. Due to the fact that the Julia set J is completely invariant under f, which means here $f^{-1}(J) = J$, and $f(J) \subset J \subset f(J) \cup \{a\}$, where a is a possible omitted value (only in the case of an entire function), it is seen that J is $f^{\circ n}$-saturated for all n and that $J \setminus \{f^{\circ k}(a) : k = 0, \dots, n-1\} \subset f^{\circ n}(J)$. Since J is perfect, the closure of $\liminf f^{\circ n}(J)$ equals J_∞, the closure of J in $\mathbb{C}_\infty$. Lemma 3.8 and Theorem 4.1 show that $\mathcal{F}|_J$, $\mathcal{F} := \{f^{\circ n} : n \in \mathbb{N}\}$, is topologically weak-mixing with respect to J_∞. According to Remark 3.1.2, this implies that $\{f^{\circ n}|_K : n \in \mathbb{N}\}$ is dense in $C(K, J_\infty)$ for generically many $K \in \mathcal{K}(J)$ (cf. also [4, Theorem 1] and [3, Theorem 2.1]).

In the case of a polynomial f it is known that for all non-empty relatively open sets $W \subset J$ eventually $f^{\circ n}(W) = J$ holds. In particular, f is topologically mixing with respect to J. Consider f of degree d having a Siegel disk F with fixed point $w_0 \in F$ and so that the boundary $\partial F \subset J$ of F is a Jordan curve. According to Carathéodory's theorem, each inverse Riemann mapping $\varphi : \mathbb{D} \to F$ with $\varphi(0) = w_0 \in F$ induces a homeomorphism (also denoted by φ) from $\mathbb{T}$ to ∂F. Moreover, $\varphi : \mathbb{T} \to \partial F$ conjugates $f|_{\partial F}$ to an irrational rotation (see e.g. [6, Theorem II 6.4] or [20, pp. 80]). We put $A := J \setminus \partial F$. Since f is d-to-one on J we have $J = f(A)$ and also $\mathcal{F}|_A$ is topologically mixing with respect to J. While A is not f-saturated, it is $\mathcal{F}$-saturated at all $z \in A$. Moreover, while $\mathcal{F}|_{\partial F}$ is hereditarily non-normal on ∂F, it is not topologically weak-mixing with respect to ∂F (since $f|_{\partial F}$ is conjugate to an irrational rotation). This shows that some saturation condition as in Theorem 4.1 has to be imposed.

Finally, we are going to consider topological versions of limsup and liminf. For a sequence (B_n) of closed sets in $\mathbb{C}_\infty$ we write $T\liminf B_n$ for the topological liminf of the sequence, i.e. the set of all $w \in \mathbb{C}_\infty$ with the property that each neighbourhood of w meets B_n for cofinitely many n, and $T\limsup B_n$ for the topological limsup, i.e. the set of all $w \in \mathbb{C}_\infty$ with the property that each neighbourhood of w meets B_n for infinitely many n (see e.g. [15], p. 25). Then $T\liminf B_n$ and $T\limsup B_n$ are closed sets in $\mathbb{C}_\infty$.

Theorem 4.4. *Let $\Omega \subset \mathbb{C}$ be a domain and $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$ a family in $M(\Omega)$. Suppose that $A \subset \Omega$ is compact and such that A is $\mathcal{F}$-saturated at all points $z \in A$.*

(1) *If for each open set $U \subset \Omega$ that meets A and each point $w \in B_* := T\liminf f_n(A)$ there is a neighbourhood V of w such that $V \subset f_n(U)$ for infinitely many n, then $\mathcal{F}|_A$ is topologically transitive with respect to B_*.*

(2) *If for each open set $U \subset \Omega$ that meets A and each point $w \in B^* := T\limsup f_n(A)$ there is a neighbourhood V of w such that $V \subset f_n(U)$ for cofinitely many n, then $\mathcal{F}|_A$ is topologically weak-mixing with respect to B.*

Proof. 1. Let $W \subset A$ be relatively open and non-empty. Then an open set $U \subset \Omega$ exists with $U \cap A \subset W$ and so that $f(A) \cap f(U) = f(A \cap U)$ for all $f \in \mathcal{F}$. Moreover, let $w \in B_*$ and let $V \subset \mathbb{C}$ be open with $w \in V$ and $V \subset f_n(U)$ for infinitely many n.

According to the definition of the topological liminf, there is some m with $V \cap f_k(A)$ non-empty for all $k \geq m$. If $n \geq m$ is so that $V \subset f_n(U)$ and if $w_n \in V \cap f_n(A)$, then also $w_n \in f_n(U)$. We have

$$w_n \in f_n(U) \cap f_n(A) = f_n(U \cap A) \subset f_n(W).$$

This shows that $\mathcal{F}$ is topologically transitive with respect to B_*,

2. Let $W \subset A$ be relatively open and non-empty, and let $U \subset \Omega$ with $U \cap A \subset W$ and $f(A) \cap f(U) = f(A \cap U)$ for all $f \in \mathcal{F}$. If $w \in B^*$, an open set $V \subset \mathbb{C}$ with $w \in V$ and m exist such that $V \subset f_n(U)$ for all $n \geq m$. According to the definition of the topological limsup, there is some $n \geq m$ such that $V \cap f_n(A)$ is non-empty. If $w_n \in V \cap f_n(A)$, then also $w_n \in f_n(U)$. Now, the proof proceeds as in part 1. □

Example 4.5. We consider the family $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ of polynomials given by $f_n(z) := (1 - 1/n)z^n$. Then $A := \mathbb{T}$ is f_n-saturated for all n and

$$T \limsup f_n(\mathbb{T}) = T \liminf f_n(\mathbb{T}) = \mathbb{T}$$

(while $\liminf f_n(\mathbb{T}) = \varnothing$). Moreover, for each open set $U \subset \mathbb{C}$ that meets $\mathbb{T}$, there is a ring domain $V = \{w : r < |w| < R\}$ with $r < 1 < R$ and such that $V \subset f_n(U)$ for all sufficiently large n. According to Theorem 4.4, $\mathcal{F}|_A$ is topologically weak-mixing with respect to $\mathbb{T}$. Remark 3.1.2 shows that $\omega(\mathcal{F}, \mathbb{T}) = C(K, \mathbb{T})$ for generically many $K \in \mathcal{K}(\mathbb{T})$.

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