Product measurability, parameter integrals, and a Fubini counterexample

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Abstract

Negative results: 1) A convolution $\int g(\cdot - y)h(y)\,dy$ need not be measurable with respect to the σ–algebra generated by the translates of g. 2) There exist a Borel set $A \subset \mathbb{R}$ and two σ–finite measures μ, ν such that $\int\int 1_A(x+y)\,d\mu(x)d\nu(y) \neq \int\int 1_A(x+y)\,d\nu(y)d\mu(x)$.

Positive result: A function of two variables, measurable with respect to a product σ–algebra $\mathcal{A}\otimes\mathcal{B}$ and partially measurable with respect to $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$, is $\mu\otimes\nu$–almost measurable with respect to $\mathcal{A}_0\otimes\mathcal{B}_0$, for μ, ν σ–finite measures on $\mathcal{A}$, $\mathcal{B}$.

1 Introduction

Let $(\mathcal{Y}, \mathcal{B}, \nu)$ be a σ–finite measure space, let $\mathcal{X}$ be a set, and let $f : \mathcal{X}\times\mathcal{Y} \to [0,\infty]$ be a function with $f(x,\cdot)$ $\mathcal{B}$–measurable for every $x \in \mathcal{X}$. Then

$$F(x) \;:=\; \int_{\mathcal{Y}} f(x,y)\,d\nu(y) \qquad (x\in\mathcal{X}) \tag{1}$$

defines a function $F : \mathcal{X} \to [0,\infty]$. Now let $\mathcal{A}_0$ be a σ–algebra on $\mathcal{X}$, and assume that $f(\cdot, y)$ is $\mathcal{A}_0$–measurable for every $y \in \mathcal{Y}$. Does it follow that F is $\mathcal{A}_0$–measurable? Surprisingly, most books on measure and integration ignore this question. Regrettably, the answer is no. Already Sierpiński (1920) provided a counterexample. His construction uses the axiom of choice and the continuum hypothesis. Without using these or similar axioms, we present below in paragraph 2.1 a simple example of a nonmeasurable F, with the right hand side of (1) being of convolution type. On the other hand, Theorem 3.1 contains a positive result, almost yielding $\mathcal{A}_0$–measurability of F under additional assumptions.

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Now assume that we are also given a σ–finite measure μ on $(\mathcal{X}, \mathcal{A}_0)$. Let us further assume that the function F from (1) is $\mathcal{A}_0$–measurable and, similarly, that $G := \int_{\mathcal{X}} f(x, \cdot)\, d\mu(x)$ is $\mathcal{B}$–measurable. Does it then follow that the "Fubini identity" $\int_{\mathcal{Y}} G(y)\, d\nu(y) = \int_{\mathcal{X}} F(x)\, d\mu(x)$ holds? Again, the answer is no, as Sierpiński (1920) remarked, essentially by specializing his construction mentioned above. This counterexample has found its way in a number of books, for example Rudin (1987) and Royden (1988), as showing that the assumption of measurability of f with respect to the product σ–algebra $\mathcal{A}_0 \otimes \mathcal{B}$ in the Fubini theorem is not superfluous. In its construction the axiom of choice is still used. The continuum hypothesis is needed only if one insists on specifying the measure spaces, for example as Lebesgue measure. That something beyond the axiom of choice is really needed in the latter case has been proved by Friedman (1980). Below we give, without using the axiom of choice or the continuum hypothesis, a simple construction of a Borel set $A \subset \mathbb{R}$ and of two σ–finite measures μ and ν, defined on suitable σ–algebras on $\mathbb{R}$, such that

$$(2) \qquad \int_{\mathbb{R}} \left[\int_{\mathbb{R}} 1_A(x+y)\, d\mu(x) \right] d\nu(y) \ \neq \ \int_{\mathbb{R}} \left[\int_{\mathbb{R}} 1_A(x+y)\, d\nu(y) \right] d\mu(x),$$

with both iterated integrals existing.

2 Nonmeasurability and a Fubini counterexample

2.1 A nonmeasurable convolution. In this section, we show that a convolution

$$(3) \qquad F \ := \ \int_{\mathbb{R}} g(\cdot - y) h(y)\, dy,$$

with g being a nonnegative bounded Borel function and h nonnegative continuous with compact support, need not be measurable with respect to

$$(4) \qquad \mathcal{A}_0 \ := \ \sigma(\{g(\cdot - y) \,:\, y \in \mathbb{R}\}),$$

the σ–algebra generated by the translates of g. This yields in particular a counterexample to the measurability of F from (1), with $f(x, y) = g(x - y)h(y)$, $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $\mathcal{A} = \mathcal{A}_0$ from (4), $\mathcal{B} = \mathcal{B}(\mathbb{R}) :=$ Borel σ–algebra on $\mathbb{R}$, and $\nu = \boldsymbol{\lambda} :=$ Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

The construction becomes clearer if we first drop the nonnegativity and boundedness conditions imposed on g, for which the necessary modifications are indicated afterwards.

Remember that a set $A \subset \mathbb{R}$ is called *meager* [or of the first category] if there is a sequence of closed and nowhere dense sets $F_n \subset \mathbb{R}$ with $A \subset \bigcup_{n \in \mathbb{N}} F_n$. Correspondingly, a set $B \subset \mathbb{R}$ is called *comeager* if its complement B^c is meager, which is equivalent to the existence of a sequence of dense open sets $G_n \subset \mathbb{R}$ with $B \supset \bigcap_{n \in \mathbb{N}} G_n$. It is easily checked that

$$(5) \qquad \mathcal{A} \ := \ \{A \in \mathcal{B}(\mathbb{R}) \,:\, A \text{ meager or comeager}\}$$

is a σ–algebra on $\mathbb{R}$. By Baire's theorem, every comeager set is dense in $\mathbb{R}$. [We have claimed in the introduction not to use the axiom of choice in constructing this example and the one in 2.2. So we have to note here that we are applying Baire's theorem only in

$\mathbb{R}$, a *separable* complete metric space, where no form of the axiom of choice is needed in its proof. Compare Oxtoby (1980), page 95.] It follows that, for example, the set $[0,\infty[$ is neither meager nor comeager. Hence we surely have the strict inclusion

$$\mathcal{A} \;\subsetneq\; \mathcal{B}(\mathbb{R}). \tag{6}$$

Now choose $A \in \mathcal{A}$ meager with $\boldsymbol{\lambda}(A^c) = 0$, for example as in Oxtoby (1980), pages 4–5. Put

$$g(x) \;:=\; x1_A(x) \qquad (x \in \mathbb{R}), \tag{7}$$

$$h(x) \;:=\; (1-|x|)_+ \qquad (x \in \mathbb{R}), \tag{8}$$

and define F as in (3) and $\mathcal{A}_0$ as in (4). Then

$$\mathcal{A}_0 \;\subset\; \mathcal{A}, \tag{9}$$

because every $g(\cdot - y)$ is Borel and vanishes on the comeager set $(y+A)^c$, and is hence $\mathcal{A}$–measurable. On the other hand, since $\boldsymbol{\lambda}(A^c) = 0$ and $\int h\,dy = 1$, $\int yh(y)\,dy = 0$,

$$F(x) \;=\; \int_{\mathbb{R}} (x-y)h(y)\,dy \;=\; x \qquad (x \in \mathbb{R}). \tag{10}$$

Hence $\sigma(F)$, the σ–algebra generated by F, is just $\mathcal{B}(\mathbb{R})$, and by (6), (9) it follows that F is not $\mathcal{A}_0$–measurable.

To obtain that same conclusion for a nonnegative and bounded g, we may replace g from (7) by $g(x) := (\pi/2 + \arctan x)1_A(x)$. Instead of calculating F explicitely, we then argue that F is still strictly increasing, and this suffices to deduce that $\sigma(F) = \mathcal{B}(\mathbb{R})$.

2.2 A Fubini counterexample. In this section, we give an example of (2). Let $\mathcal{A}$ be as in (5) and define $\mu|\mathcal{A}$ by

$$\mu(A) \;:=\; \begin{cases} 0 & \text{if } A \text{ is meager,} \\ 1 & \text{if } A \text{ is comeager.} \end{cases} \tag{11}$$

This is possible, since no set $A \subset \mathbb{R}$ is simultaneously meager and comeager, for otherwise $\emptyset = A \cap A^c$ would be comeager, in contradiction to Baire's theorem. It is easy to check that μ is a probability measure on $(\mathbb{R}, \mathcal{A})$. Let again $\nu := \boldsymbol{\lambda} :=$ Lebesgue measure on $\mathcal{B} := \mathcal{B}(\mathbb{R})$, and choose $A \in \mathcal{A}$ meager with $\boldsymbol{\lambda}(A^c) = 0$. Then $1_A(\cdot + y)$ is $\mathcal{A}$–measurable with

$$\int_{\mathbb{R}} 1_A(x+y)\,d\mu(x) \;=\; \mu(A-y) \;=\; 0 \qquad (y \in \mathbb{R}).$$

On the other hand, we have

$$\int_{\mathbb{R}} 1_A(x+y)\,d\nu(y) \;=\; \boldsymbol{\lambda}(A-x) \;=\; \infty \qquad (x \in \mathbb{R}).$$

Hence (2) is obviously true in this case.

3 Measurability

Here is a positive result, having a certain measurability property of F from (1) among its conclusions. An application of this will occur in Mattner (1999).

3.1 Theorem. *Let $(\mathcal{X},\mathcal{A},\mu)$ and $(\mathcal{Y},\mathcal{B},\nu)$ be σ–finite measure spaces, let $f:\mathcal{X}\times\mathcal{Y}\to[0,\infty]$ be a function measurable with respect to the product σ–algebra $\mathcal{A}\otimes\mathcal{B}$, and put*

$$\begin{aligned} \mathcal{A}_0 &:= \sigma(\{f(\cdot,y) \,:\, y\in\mathcal{Y}\}),\\ \mathcal{B}_0 &:= \sigma(\{f(x,\cdot) \,:\, x\in\mathcal{X}\}),\\ \overline{\mathcal{A}_0} &:= \{A\in\mathcal{A} \,:\, \exists\, A_0\in\mathcal{A}_0 \text{ with } A=A_0\,[\mu]\},\\ \overline{\mathcal{B}_0} &:= \{B\in\mathcal{B} \,:\, \exists\, B_0\in\mathcal{A}_0 \text{ with } B=B_0\,[\nu]\},\\ \overline{\mathcal{A}_0\otimes\mathcal{B}_0} &:= \{C\in\mathcal{A}\otimes\mathcal{B} \,:\, \exists\, C_0\in\mathcal{A}_0\otimes\mathcal{B}_0 \text{ with } C=C_0\,[\mu\otimes\nu]\}. \end{aligned}$$

Then f is $\overline{\mathcal{A}_0\otimes\mathcal{B}_0}$–measurable, $\int_{\mathcal{Y}} f(\cdot,y)\,d\nu(y)$ is $\overline{\mathcal{A}_0}$–measurable, and $\int_{\mathcal{X}} f(x,\cdot)\,d\mu(x)$ is $\overline{\mathcal{B}_0}$–measurable.

Here and in what follows, we write $A=A_0\,[\mu]$ for $\mu(A\triangle A_0)=0$. Below we also use the corresponding notation $f=g\,[\mu]$ for functions, meaning $\mu(\{x : f(x)\neq g(x)\})=0$.

3.2 Remarks. Let us retain the notation and assumptions of Theorem 3.1.

a) The parameter integral $\int_{\mathcal{Y}} f(\cdot,y)\,d\nu(y)$ need not be $\mathcal{A}_0$–measurable and f need not be $\mathcal{A}_0\otimes\mathcal{B}_0$–measurable, as the example in 2.1 shows.

b) The function f need not be $\overline{\mathcal{A}_0}\otimes\overline{\mathcal{B}_0}$–measurable. As an example proving this remark, we may take $(\mathcal{X},\mathcal{A},\mu):=(\mathcal{Y},\mathcal{B},\nu):=([0,1],\mathcal{B}([0,1]),\boldsymbol{\lambda}^1)$, $D:=\{(x,x) \,:\, x\in[0,1]\}$, and $f:=1_D$. [We now write $\boldsymbol{\lambda}^d$ for d–dimensional Lebesgue measure.] Then

$$\begin{aligned} \mathcal{A}_0=\mathcal{B}_0 &= \{A\in\mathcal{B}([0,1]) \,:\, A \text{ countable or cocountable}\},\\ \overline{\mathcal{A}_0}=\overline{\mathcal{B}_0} &= \{A\in\mathcal{B}([0,1]) \,:\, \boldsymbol{\lambda}^1(A)\in\{0,1\}\}, \end{aligned}$$

and we claim that f is not $\overline{\mathcal{A}_0}\otimes\overline{\mathcal{B}_0}$–measurable. To prove this, put

$$\mathcal{C} \;:=\; \Big\{C\in\mathcal{B}([0,1]^2) \,:\, \Big(\boldsymbol{\lambda}^2(C),\textstyle\int_0^1 1_C(x,x)\,d\boldsymbol{\lambda}^1(x)\Big)\;\in\;\{(0,0),(1,1)\}\Big\}.$$

Then $\mathcal{C}$ is a σ–algebra containing $\big\{A\times B \,:\, A\in\overline{\mathcal{A}_0},\, B\in\overline{\mathcal{B}_0}\big\}$, and hence satisfies $\overline{\mathcal{A}_0}\otimes\overline{\mathcal{B}_0}\subset\mathcal{C}$. But $D\notin\mathcal{C}$, so that $D\notin\overline{\mathcal{A}_0}\otimes\overline{\mathcal{B}_0}$.

c) Let us write more explicitly $\overline{\mathcal{A}_0}(\mu)$ in place of $\overline{\mathcal{A}_0}$. From Theorem 3.1, we may deduce the measurability of $F:=\int f(\cdot,y)\,d\nu(y)$ with respect to $\bigcap_\mu \overline{\mathcal{A}_0}(\mu)$, the intersection being over all μ as in the theorem. This, however, must not be confused with the more restrictive property of universal $\mathcal{A}_0$–measurability of F [see Cohn (1980), pages 280–283, for the definition and for illuminating facts]. Indeed, our measures μ are supposed to be defined on $\mathcal{A}$, and not merely on $\mathcal{A}_0$ or its μ–completion. For example, in the situation of 2.1, one can use the measure μ from (11) to deduce that the σ–algebra of all universally $\mathcal{A}_0$–measurable is contained in $\widetilde{\mathcal{A}_0}:=\{A\subset\mathbb{R} \,:\, A \text{ meager or comeager}\}$. Since $\widetilde{\mathcal{A}_0}$ differs from $\mathcal{A}_0$ only by non–Borel sets, we see that F from (3), (7), (8) is not universally $\mathcal{A}_0$–measurable. By the way, the known fact that μ from (11) can not be extended to a measure on $\mathcal{B}(\mathbb{R})$ [see Oxtoby (1980), page 86] follows from our present considerations, since otherwise we would have $\overline{\mathcal{A}_0}(\mu) = \widetilde{\mathcal{A}_0} \cap \mathcal{B}(\mathbb{R}) = \mathcal{A}_0$, and Theorem 3.1 would yield $\mathcal{A}_0$–measurability of F.

3.3 Proof of the Theorem 3.1. Obvious arguments show that we may assume in addition that

$$\mu, \nu \text{ are finite and } f \text{ is bounded.} \tag{12}$$

The proof of the theorem splits into two parts as follows.

Claim 1. *Under the assumptions of the theorem and* (12),

$$F \ := \ \int f(\cdot, y)\, d\nu(y) \tag{13}$$

is $\overline{\mathcal{A}_0}$*–measurable.*

Proof: Let us first recall the "mean value theorem" for vector valued integration: Let E be a topological vector space, $(\Omega, \mathcal{A}, \mu)$ be a measure space, and $g : \Omega \to E$ be a function. Then an $x \in E$ is called the weak (or Pettis) integral of g, and we write $\int g\, d\mu := x$, if i) the dual space E' of E separates points on E, ii) the scalar function $\langle y, g(\cdot)\rangle$ belongs to $\mathcal{L}^1(\Omega, \mathcal{A}, \mu)$ for every $y \in E'$, and iii) $\int \langle y, g(\omega)\rangle\, d\mu(\omega) = \langle y, x\rangle$ for every $y \in E'$. [This is the definition adopted by Edwards (1965), p. 566, and by Rudin (1991), p. 77.] If now E is in particular locally convex Hausdorff and μ is bounded, then the weak integral, if it exists, necessarily satisfies

$$\int g\, d\mu \ \in \ \mu(\Omega) \cdot \overline{\operatorname{conv}}\, g(\Omega), \tag{14}$$

with $\overline{\operatorname{conv}}$ indicating convex closure. This "mean value theorem" is surely well known. It follows easily from the Hahn–Banach theorem: Apply Theorem 3.4 (b) of Rudin (1991) to $A := \left\{\int g\, d\mu\right\}$ and $B := \overline{\operatorname{conv}}\, g(\Omega)$.

We now start with the proof proper. The functions $f(\cdot, y) : \mathcal{X} \to \mathbb{R}$, as well as F from (13), are $\mathcal{A}$–measurable [by $\mathcal{A} \otimes \mathcal{B}$–measurability of f and by Fubini] and bounded, and hence belong to $\mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$. Let $[f(\cdot, y)], [F] \in L^1(\mathcal{X}, \mathcal{A}, \mu)$ denote their corresponding equivalence classes. We claim that

$$[F] \ = \ \int_{\mathcal{Y}} [f(\cdot, y)]\, d\nu(y), \tag{15}$$

in the weak sense recalled above, applied to the Banach space $E = L^1(\mathcal{X}, \mathcal{A}, \mu)$ with dual space $L^\infty(\mathcal{X}, \mathcal{A}, \mu)$. To prove this, let $h \in [h] \in L^\infty(\mathcal{X}, \mathcal{A}, \mu)$. An obvious Fubini calculation, using the definition of F and the $\mathcal{A} \otimes \mathcal{B}$–measurability of f, yields

$$\langle [h], [F] \rangle \ = \ \int_{\mathcal{X}} h(x) F(x)\, d\mu(x) \ = \ \int_{\mathcal{Y}} \langle [h], [f(\cdot, y)] \rangle\, d\nu(y),$$

which confirmes (15). [Actually, (15) is even true with the right hand side read as a Bochner integral, but we do not need this fact here.] We now use that each $f(\cdot,y)$ is $\mathcal{A}_0$–measurable, where of course $\mathcal{A}_0 \subset \mathcal{A}$. This implies that the function $y \mapsto [f(\cdot,y)]$ takes its values in

$$S \;:=\; \big\{\Phi \in L^1(\mathcal{X},\mathcal{A},\mu) \,:\, \exists\, \mathcal{A}_0\text{–measurable } \varphi \in \Phi\big\},$$

which is easily seen to be a closed subspace of $L^1(\mathcal{X},\mathcal{A},\mu)$. The mean value theorem (14) now yields $[F] \in S$, which is the desired conclusion. ∎

Claim 2. *Under the assumptions of the theorem and (12), and assuming the truth of Claim 1, f is $\overline{\mathcal{A}_0 \otimes \mathcal{B}_0}$–measurable.*

Proof: We consider the restrictions

$$\overline{\mu_0} := \mu|\overline{\mathcal{A}_0}, \qquad \overline{\nu_0} = \nu|\overline{\mathcal{A}_0},$$

and define a function $\tau : \overline{\mathcal{A}_0} \otimes \overline{\mathcal{B}_0} \to [0,\infty]$ by

$$(16) \qquad \tau(C) \;:=\; \int_{\mathcal{X}}\int_{\mathcal{Y}} f(x,y)1_C(x,y)\,d\overline{\nu_0}(y)\,d\overline{\mu_0}(x) \qquad (C \in \overline{\mathcal{A}_0} \otimes \overline{\mathcal{B}_0}),$$

and we emphasize that the right hand side has to be read as an iterated integral. In order to show its existence, we have to check that the function $x \mapsto \int_{\mathcal{Y}} f(x,y)1_C(x,y)\,d\overline{\nu_0}(y)$ is $\overline{\mathcal{A}_0}$–measurable. For the special case $C = A \times B$ with $A \in \overline{\mathcal{A}_0}$ and $B \in \overline{\mathcal{B}_0}$, this follows from Claim 1, applied to $\overline{\mathcal{A}_0}$ in place of $\mathcal{A}_0$ and $f(x,y)1_B(y)$ in place of $f(x,y)$, and using $\overline{\overline{\mathcal{A}_0}} = \overline{\mathcal{A}_0}$. The general case follows as usual via Sierpiński's lemma [Satz I.6.8 in Elstrodt (1996)]. Thus τ is well–defined. It is easily checked that τ is a measure, and that every set of $\overline{\mu_0}\otimes\overline{\nu_0}$–measure zero is of τ–measure zero as well. Hence the Lebesgue–Radon–Nikodym theorem yields the existence of an $\overline{\mathcal{A}_0} \otimes \overline{\mathcal{B}_0}$–measurable function $\tilde{f} : \mathcal{X} \times \mathcal{Y} \to [0,\infty]$ such that

$$\tau(C) \;=\; \int_C \tilde{f}\, d\overline{\mu_0} \otimes \overline{\nu_0} \qquad (C \in \overline{\mathcal{A}_0} \otimes \overline{\mathcal{B}_0}).$$

By (16) and Fubini, this implies in particular

$$(17) \quad \int_{A_0}\int_{B_0} f(x,y)\,d\overline{\nu_0}(y)\,d\overline{\mu_0}(x) \;=\; \int_{A_0}\int_{B_0} \tilde{f}(x,y)d\overline{\nu_0}(y)\,d\overline{\mu_0}(x) \qquad (A_0 \in \overline{\mathcal{A}_0}, B_0 \in \overline{\mathcal{B}_0}).$$

Since, using (12), both sides in (17) are always finite, we may conclude for every $B_0 \in \overline{\mathcal{B}_0}$:

$$\int_{B_0} f(\cdot,y)\,d\overline{\nu_0}(y) \;=\; \int_{B_0} \tilde{f}(\cdot,y)\,d\overline{\nu_0}(y) \qquad [\overline{\mu_0}].$$

Trivially, this remains true if $[\overline{\mu_0}]$ is replaced by $[\mu]$, and an integration yields

$$(18) \qquad \int_{A}\int_{B_0} f(x,y)\,d\overline{\nu_0}(y)\,d\mu(x) \;=\; \int_{A}\int_{B_0} \tilde{f}(x,y)d\overline{\nu_0}(y)\,d\mu(x) \qquad (A \in \mathcal{A}, B_0 \in \overline{\mathcal{B}_0}).$$

We now want to interchange the order of integrations. Since $\tilde{f}$ is trivially $\mathcal{A} \otimes \overline{\mathcal{B}_0}$–measurable, we may obviously do this on the right hand side of (18). To do the same on the left hand side, we rewrite it successively as

$$\int_A \int_{B_0} f(x,y)\, d\nu(y)\, d\mu(x) = \int_{B_0} \int_A f(x,y)\, d\mu(x)\, d\nu(y) = \int_{B_0} \int_A f(x,y)\, d\mu(x)\, d\overline{\nu_0}(y),$$

where the last equality follows from a second application of Claim 1, with the role of the variables interchanged. Thus (18) yields

$$\int\limits_{B_0} \int\limits_A f(x,y)\, d\mu(x)\, d\overline{\nu_0}(y) \;=\; \int\limits_{B_0} \int\limits_A \tilde{f}(x,y) d\mu(x)\, d\overline{\nu_0}(y) \qquad (A \in \mathcal{A}, B_0 \in \overline{\mathcal{B}_0}). \tag{19}$$

Now the argument leading from (17) to (18) can be repeated to lead from (19) to a corresponding statement with B in place of B_0, ν in place of $\overline{\nu_0}$, and $\mathcal{B}$ in place of $\overline{\mathcal{B}_0}$, which is equivalent to

$$\int_{A\times B} f\, d\mu \otimes \nu \;=\; \int_{A\times B} \tilde{f}\, d\mu \otimes \nu \qquad (A \in \mathcal{A},\ B \in \mathcal{B}).$$

This shows that $f = \tilde{f}\,[\mu \otimes \nu]$, which yields the desired conclusion. ∎

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