# On Burdick's symmetry problem

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*Dedicated to Professor Johann Pfanzagl, on the occasion of his seventieth birthday*

**Abstract**

Let $P$ be a probability measure on the real line $\mathbb{R}$ such that each of the product measures $P^{\otimes n}$ assigns the value $1/2$ to every half space in $\mathbb{R}^n$ having the origin as a boundary point. Then $P$ is symmetric.

Example: A strictly stable law on $\mathbb{R}$ is symmetric iff it has median zero.

The treated symmetry problem is related to the problem of characterizing the distribution of $X_1$ by the distribution of $(X_2+X_1,\ldots,X_n+X_1)$, with $X_1,\ldots,X_n$ being independent and identically distributed random variables.

# 1 Introduction and main result

**1.1 Burdick's problem.** Let $P$ be a probability measure on the real line $\mathbb{R}$, let $n \in \mathbb{N} = \{1,2,\ldots\}$, and assume that the $n$–fold product measure $P^{\otimes n}$ assigns the value $1/2$ to every half space in $\mathbb{R}^n$ having the origin as a boundary point. Does it follow that $P$ is symmetric?

To make this question precise, and reasonable also for discontinuous probability measures, let $\mathcal{H}_n$ be the set of all "normalized signed indicator functions" of the said half spaces: A function $h$ on $\mathbb{R}^n$ belongs to $\mathcal{H}_n$ iff

$$(1) \qquad h(x) \;=\; \operatorname{sgn}(\alpha x) \qquad (x \in \mathbb{R}^n)$$

for some $\alpha \in \mathbb{R}^n \setminus \{0\}$, where $\alpha x := \alpha_1 x_1 + \ldots + \alpha_n x_n$ and $\operatorname{sgn}(t) := 1(t>0) - 1(t<0)$. Let further $\operatorname{Prob}(\mathbb{R})$ denote the set of all probability measures on $\mathbb{R}$ and consider the following properties a $P \in \operatorname{Prob}(\mathbb{R})$ might have:

(S) $P$ is symmetric with respect to sign change,

($\mathrm{B}_n$) $P^{\otimes n}(h) = 0$ for every $h \in \mathcal{H}_n$.

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We obviously have the implications

$$(\mathrm{B}_1) \Leftarrow (\mathrm{B}_2) \Leftarrow \ldots \Leftarrow (\mathrm{S}). \tag{2}$$

It is obvious as well that condition $(\mathrm{B}_1)$, which roughly speaking says that $P$ has median zero, is to weak to imply (S). So one may ask whether any of the conditions $(\mathrm{B}_n)$ with $n \geq 2$ is strong enough to imply (S).

This problem was apparently first considered as well as partially solved by Burdick (1972). He showed that condition $(\mathrm{B}_2)$ does imply (S) in the presence of the two auxiliary conditions

(AC) $P$ is absolutely continuous with respect to Lebesgue measure,

(FM) $\int_{\mathbb{R}\setminus\{0\}} |x|^\varepsilon \, dP(x) < \infty$ for some $\varepsilon \in \mathbb{R} \setminus \{0\}$,

and he asked whether the fractional moment condition (FM) can be omitted. Feuerverger & Steele (1985) gave a negative answer.

In the present paper, we answer Burdick's question in the negative also with the stronger condition $(\mathrm{B}_n)$ (for any *fixed* $n \geq 3$ ) in place of $(\mathrm{B}_2)$. We also show that the condition (AC) can be omitted in Burdick's theorem. And, in achieving our main goal of obtaining a version of Burdick's theorem not containing any technical assumptions like (FM), we prove the claim indicated in the abstract of this paper. To sum up:

**1.2 Theorem.**
*a) The conditions* $(\mathrm{B}_n)$ *for* $n \in \mathbb{N}$ *jointly imply* (S).
*b) The conditions* $(\mathrm{B}_2)$ *and* (FM) *jointly imply* (S).
*c) For each* $n \in \mathbb{N}$, $(\mathrm{B}_n)$ *does not imply* (S).

**1.3 Organization of this paper.** Section 2 below contains supplementary information concerning the assumptions of Theorem 1.2 [in 2.1 to 2.3], an application to stable laws [in 2.4], and a discussion of the relation of Burdick's problem with the problem of charaterizing distributions via certain linear statistics [in 2.5].

Section 3 contains proofs of Theorem 1.2 and of the claims in Subsection 2.3. The main work is contained in Proposition 3.2 and its proof.

# 2 Remarks and complements

**2.1 "Counterexamples".** Three results complementing part c) of Theorem 1.2 in showing that certain variants of the conditions $(\mathrm{B}_n)$ are not sufficient for deducing (S) are contained in Example 2 of Freedman & Diaconis (1982), in Kochar (1992), and in Propositions 1 and 2 of Mattner (199?).

**2.2 Sufficient conditions for the validity of** (FM). Condition (FM) is always true when $P$ has a Lebesgue density bounded near zero, or when $P$ is a lattice distribution. [Take $\varepsilon = -1/2$.] Hence, for such $P$, condition $(\mathrm{B}_2)$ implies (S). [In the first–mentioned case, this follows already from Burdick's paper, which could be overlooked due to the way he formulated condition (FM).]

**2.3 Possible weakenings of** (FM). It is possible to replace condition (FM) in part b) of Theorem 1.2 by similar but weaker conditions, for example by

(FM$_+$) $\int_{]0,\infty[} x^\varepsilon \, dP(x) < \infty$ for some $\varepsilon \in \mathbb{R} \setminus \{0\}$.

[Our proof of the theorem given below uses only this weaker assumption.] However, as then follows from the validity of part b) of the theorem with (FM$_+$) in place of (FM), these two conditions are actually equivalent in the presence of condition (B$_2$). [Reason: If $P$ satisfies (S), then $\int_{]-\infty,0[}(-x)^\varepsilon \, dP(x) = \int_{]0,\infty[} x^\varepsilon \, dP(x)$.]

On the other hand, finiteness of all logarithmic moments, that is the condition $\int_{\mathbb{R}\setminus\{0\}} |\log|x||^n \, dP(x) < \infty$ for every $n \in \mathbb{N}$, is not a sufficient substitute for (FM). This claim, proved at the end of Section 3, answers a question of Feuerverger and Steele (1985).

**2.4 An application to stable laws.** Let $P \in \mathrm{Prob}(\mathbb{R})$ be strictly stable [in the sense of Feller (1971), page 170, or Zolotarev (1986), page 6], and let $P$ have median zero. By the continuity of the nondegenerate stable laws, the latter assumption is equivalent to (B$_1$). By strict stability, this implies (B$_n$) for every $n \in \mathbb{N}$. Hence part a) of Theorem 1.2 yields symmetry of $P$.

CONCLUSION: *A strictly stable law is symmetric iff it has median zero.*

This can also be proved computationally, in the case of exponent $\alpha \neq 1$ by referring to formula (2.2.30) of Zolotarev (1986), and in the case of exponent $\alpha = 1$ by observing that then the only strictly stable laws are shifts of the centered Cauchy and Dirac measures. So the conclusion could be regarded as known, but it is perhaps interesting to observe that our proof given above does not depend in any way on the representation theorem for the stable laws.

See Shkolnik (1993) for the related problem of approximating the median of an arbitrary stable law.

**2.5 The relation with the characterization of parent distributions by the distribution of lower dimensional linear statistics.** Let $X = (X_1, \ldots, X_n)$ denote the identity function on $\mathbb{R}^n$, let, for $n \geq 2$,

$$S^{(n)} \;=\; (X_2 + X_1, \ldots, X_n + X_1),$$

and consider for any pair $(\mu, \nu)$ of sub–probability measures [nonnegative measures with total mass at most one] on $\mathbb{R}$ the condition that the distributions of $S^{(n)}$ with respect to the product measures $\mu^{\otimes n}$ and $\nu^{\otimes n}$ are identical:

(C$_n^+$) $S^{(n)}\square\mu^{\otimes n} = S^{(n)}\square\nu^{\otimes n}$.

Let further $\varphi$ and $\psi$ denote the Fourier transforms of $\mu$ and $\nu$. [See Subsection 3.1 below for more precise definitions of the notation used here.] Then condition (C$_n^+$) is easily seen to be equivalent to the functional equation

($\widehat{\mathrm{C}}_n^+$) $\varphi(\sum_{k=1}^{n-1} t_k) \prod_{k=1}^{n-1} \varphi(t_k) = \psi(\sum_{k=1}^{n-1} t_k) \prod_{k=1}^{n-1} \psi(t_k) \qquad ((t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1})$.

Now a look at Proposition 3.2, where the same functional equation occurs under the label ($\widehat{\mathrm{B}}_n$), reveals:

OBSERVATION. *Let* $P \in \mathrm{Prob}(\mathbb{R})$ *and let* $(\mu, \nu)$ *be related to* $P$ *as in* (3), (4) *and* (5) *of Proposition 3.2. Then, for every* $n \geq 2$, (B$_n$) *is equivalent to* (C$_n^+$).

This result does not appear to be obvious from the outset, and I do not know how to prove it without using Fourier transforms. It would be obvious, if (B$_n$) were obviously equivalent to the symmetry of $P$ and if also ($\mathrm{C}_n^+$) were obviously equivalent to $\mu = \nu$, since symmetry of $P$ is obviously equivalent to $\mu = \nu$, but part c) of Theorem 1.2 shows that the first hypothetical equivalence [and thus, by the observation, the second as well] is not true.

We finally note that a condition similar to ($\mathrm{C}_n^+$), namely the condition ($\mathrm{C}_n^-$) which results if every plus sign in the definition of $S^{(n)}$ is changed to a minus sign, has been studied several times in the literature. [See Brown (1940) and the review in Bondesson (1983) for the case $n \geq 3$, Carnal & Dozzi (1989) and Lešanovský & Rataj (1991) for the peculiar case $n = 2$, Sasvari & Wolf (1986) for related work, and the references contained in these papers. In particular, Kovalenko (1960) contains a result analogous to Theorem 1.2 c), according to Kovalenko (1966) and Schmetterer (1967).] Similarly as above, ($\mathrm{C}_n^-$) is seen to be equivalent to the functional equation (12) below. Hence the proof of Proposition 3.2 shows that ($\mathrm{B}_n$) implies ($\mathrm{C}_n^-$). This implication is, however, much less surprising than the equivalence of ($\mathrm{B}_n$) and ($\mathrm{C}_n^+$). Indeed, to prove the said implication, the introduction of the Fourier transforms $\varphi$ and $\psi$ is easily seen to be an unnecessary detour.

# 3 Proofs

**3.1 Notation.** Let $\delta \in \mathrm{Prob}(\mathbb{R})$ denote the Dirac measure located at zero. We write $f\square\mu$ for the image measure of a measure $\mu$ under a function $f$ (distribution of $f$ with respect to $\mu$), and use the standard notation for convolutions, $\mu * \nu = ((x,y) \mapsto x+y)\square\mu\otimes\nu$, of measures $\mu, \nu$ on $\mathbb{R}$. As in the preceding formula, an expression like $f\square\mu\otimes\nu$ is to be read as $f\square(\mu\otimes\nu)$, not as $(f\square\mu)\otimes\nu$.

The Fourier transform of a bounded measure $\mu$ on $\mathbb{R}$ is defined to be the function $\mathbb{R} \ni t \mapsto \widehat{\mu}(t) := \int \exp(itx)\, d\mu(x)$. For any $\mathbb{C}$–valued function $\varphi$, $\bar{\varphi}$ denotes its complex conjugate.

**3.2 Proposition.** *Let* $P \in \mathrm{Prob}(\mathbb{R})$. *Decompose* $P$ *as*

$$P = L + c\delta + R \tag{3}$$

*with* $c = P(\{0\})$, $L = 1_{]-\infty,0[}P$, $R = 1_{]0,\infty[}P$. *Let* $\varphi = \widehat{\mu}$ *and* $\psi = \widehat{\nu}$ *be the Fourier transforms of the image measures* $\mu, \nu$ *defined by*

$$\mu = (\,]-\infty,0[\, \ni x \mapsto \log -x\,)\square L, \tag{4}$$

$$\nu = \log\square R. \tag{5}$$

*Then* (S) *is equivalent to the identity*

($\widehat{\mathrm{S}}$) $\varphi = \psi$,

*while* ($\mathrm{B}_1$) *is equivalent to*

($\widehat{\mathrm{B}}_1$) $\varphi(0) = \psi(0)$

*and, for each* $n \in \mathbb{N}$ *with* $n \geq 2$, ($\mathrm{B}_n$) *is equivalent to the functional equation*

($\widehat{\mathrm{B}}_n$) $\varphi(\sum_{k=1}^{n-1} t_k)\prod_{k=1}^{n-1}\varphi(t_k) = \psi(\sum_{k=1}^{n-1} t_k)\prod_{k=1}^{n-1}\psi(t_k) \qquad ((t_1,\ldots,t_{n-1}) \in \mathbb{R}^{n-1}).$

**Proof.** Let $P \in \mathrm{Prob}(\mathbb{R})$.

IDEAS. The simple key ideas of this proof are contained in step 5 [division and application of the Radon–Cramér–Wold theorem], in step 7 [the mutual singularity argument] and in step 8 [possibility of subsuming several functional equations under the one in $(\widehat{\mathrm{B}}_n)$].

Step 6 and the induction arguments in steps 3 and 4 handle a complication caused by the possibility of $c = P(\{0\})$ being nonzero. Unfortunately, this makes the proof look more complicated than it essentially is.

STEP 1. The equivalence of (S) and $(\widehat{\mathrm{S}})$ obviously follows from the uniqueness theorem for Fourier transforms.

STEP 2. The condition $(\widehat{\mathrm{B}}_1)$ is equivalent to $\mu(\mathbb{R}) = \nu(\mathbb{R})$, hence equivalent to $(\mathrm{B}_1)$.

STEP 3. *To prove the equivalence of the conditions* $(\mathrm{B}_n)$ *and* $(\widehat{\mathrm{B}}_n)$ *for any* $n \geq 2$*, we may assume that* $P$ *satisfies* $(\mathrm{B}_{n-1})$ *and* $(\widehat{\mathrm{B}}_{n-1})$.

PROOF. Use mathematical induction and the obvious implications $(\mathrm{B}_n) \Rightarrow (\mathrm{B}_{n-1})$ and $(\widehat{\mathrm{B}}_n) \Rightarrow (\widehat{\mathrm{B}}_{n-1})$.

STEP 4. *Let* $n \in \mathbb{N}$ *with* $n \geq 2$*, let* $P$ *satisfy* $(\mathrm{B}_{n-1})$*, let* $X^{(n)} = (X_1, \dots, X_n)$ *be the identity function on* $\mathbb{R}^n$*, and put*

$$Z^{(n)} \;:=\; \left(\frac{X_k}{X_1} \,:\, 2 \leq k \leq n\right),$$

*with the usual conventions concerning zero denominators.*

*Then* $(\mathrm{B}_n)$ *is equivalent to the condition*

$$(\mathrm{B}_{n,n-1}) \qquad Z^{(n)}\Box(R-L) \otimes (L+R)^{\otimes(n-1)} \;=\; 0.$$

PROOF. For $n \in \mathbb{N}$ with $n \geq 2$, let us consider the further conditions

$$(\mathrm{B}_{n,0}) \qquad Z^{(n)}\Box(R-L) \otimes P^{\otimes(n-1)} \;=\; 0$$

and, more generally, for $m \in \{0, \dots, n-1\}$,

$$(\mathrm{B}_{n,m}) \qquad Z^{(n)}\Box(R-L) \otimes (L+R)^{\otimes m} \otimes P^{\otimes(n-1-m)} \;=\; 0,$$

with the convention that the factors $(L+R)^{\otimes 0}$, occuring iff $m = 0$, and $P^{\otimes 0}$, occuring iff $m = n-1$, are to be ignored. Let us finally also consider the condition

$$(\mathrm{B}_{1,0}) \qquad (R-L)(\mathbb{R}) \;=\; 0.$$

By step 5 below, $(\mathrm{B}_n)$ and $(\mathrm{B}_{n,0})$ are equivalent, while in step 6 below, $(\mathrm{B}_{n,m-1})$ is shown to be equivalent to $(\mathrm{B}_{n,m})$, for every $m \in \{1, \dots, n-1\}$, assuming the truth of $(\mathrm{B}_{n-1,m-1})$. This latter assumption imposes no restriction when proving the present claim: For $n = 2$, $(\mathrm{B}_{n-1,m-1})$ can only be $(\mathrm{B}_{1,0})$, which is equivalent to $(\mathrm{B}_1)$, hence true by hypothesis. For $n > 2$, we may by mathematical induction assume the equivalence and thus, by the hypothesis $(\mathrm{B}_{n-1})$, the truth of all conditions $(\mathrm{B}_{n-1})$, $(\mathrm{B}_{n-1,0})$, $\dots$, $(\mathrm{B}_{n-1,n-2})$.

Thus steps 5 and 6 below complete the proof of the present claim.

STEP 5. $(\mathrm{B}_n)$ *is equivalent to* $(\mathrm{B}_{n,0})$.

PROOF. For $t \in \mathbb{R}$ and $(\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{R}^{n-1}$,

$$\begin{aligned} P^{\otimes n}(\operatorname{sgn}(\sum_{k=2}^n \alpha_k X_k - tX_1)) &= -L \otimes P^{\otimes(n-1)}(\operatorname{sgn}(\sum_{k=2}^n \alpha_k \frac{X_k}{X_1} - t)) \\ &\quad + c\delta \otimes P^{\otimes(n-1)}(\operatorname{sgn}(\sum_{k=2}^n \alpha_k X_k)) \\ &\quad + R \otimes P^{\otimes(n-1)}(\operatorname{sgn}(\sum_{k=2}^n \alpha_k \frac{X_k}{X_1} - t)) \qquad [\text{by}(3)] \\ &= (R-L) \otimes P^{\otimes(n-1)}(\operatorname{sgn}(\sum_{k=2}^n \alpha_k \frac{X_k}{X_1} - t)) \qquad [\text{by } (\mathrm{B}_{n-1})]. \end{aligned}$$

To see that the above calculation implies the claim, use the fact that the family of functions $\{\operatorname{sgn}(\cdot - t) : t \in \mathbb{R}\}$ is a determining class for signed measures on $\mathbb{R}$, and the Radon–Cramér–Wold theorem on the determination of measures by one–dimensional marginals.

STEP 6. *If $m \in \{1, \dots, n-1\}$ and if $(\mathrm{B}_{n-1,m-1})$ is true, then $(\mathrm{B}_{n,m-1})$ and $(\mathrm{B}_{n,m})$ are equivalent.*

PROOF. If $n = 2$, then $m = 1$ and

$$(6) \qquad Z^{(2)}\Box(R-L) \otimes P = Z^{(2)}\Box(R-L) \otimes (R+L) + Z^{(2)}\Box(R-L) \otimes c\delta.$$

The last summand in (6) is $c \cdot (R-L)(\mathbb{R}) \cdot \delta$, which is the zero measure by $(\mathrm{B}_{1,0})$. Hence (6) shows that $(\mathrm{B}_{2,0})$ and $(\mathrm{B}_{2,1})$ are equivalent.

Now let $n > 2$. Then

$$(7) \qquad \begin{aligned} &Z^{(n)}\Box(R-L) \otimes (L+R)^{\otimes(m-1)} \otimes P^{\otimes(n-m)} \\ &\quad = Z^{(n)}\Box(R-L) \otimes (L+R)^{\otimes m} \otimes P^{\otimes(n-1-m)} \\ &\quad + Z^{(n)}\Box(R-L) \otimes (L+R)^{\otimes(m-1)} \otimes c\delta \otimes P^{\otimes(n-1-m)}. \end{aligned}$$

Now the last summand in (7) is zero because its image under the permutation $(z_1, \dots, z_{n-1}) \mapsto (z_1, \dots, z_{m-1}, z_{m+1}, \dots z_{n-1}, z_m)$ is the measure

$$\big(Z^{(n-1)}\Box(R-L) \otimes (L+R)^{\otimes(m-1)} \otimes P^{(n-1-m)}\big) \otimes c\delta,$$

and this is zero by assumption $(\mathrm{B}_{n-1,m-1})$. Hence (7) shows that $(\mathrm{B}_{n,m-1})$ and $(\mathrm{B}_{n,m})$ are equivalent.

STEP 7. To continue, put for $\varepsilon \in \{-1, 1\}$

$$U_\varepsilon = \begin{cases} L & (\varepsilon = -1) \\ R & (\varepsilon = 1) \end{cases}$$

and

$$I_\varepsilon = \begin{cases} ]-\infty, 0[ & (\varepsilon = -1) \\ ]0, \infty[ & (\varepsilon = 1). \end{cases}$$

Then

$$(8) \qquad Z_n\Box(R-L) \otimes (L+R)^{\otimes(n-1)} = \sum_{\varepsilon \in \{-1,1\}^n} \operatorname{sgn}(\varepsilon(1)) \cdot Z_n\Box \bigotimes_{k=1}^n U_{\varepsilon(k)}.$$

Observe that the measure $Z_n \square \bigotimes_{k=1}^n U_{\varepsilon(k)}$ is supported by the set

$$\bigtimes_{k=2}^{n} I_{\varepsilon(k)\cdot\varepsilon(1)}.$$

Hence any two terms in the sum on the right hand side of (8) with different indices $\varepsilon'$, $\varepsilon'' \in \{0,1\}^n$ are mutually singular unless $\varepsilon'(k) \neq \varepsilon''(k)$ holds for *every* $k \in \{1, \ldots, n\}$.

It follows that condition $(\mathrm{B}_{n,n-1})$ is equivalent to a system of $2^{n-1}$ equations, of which $n$ are stated below, and the remaining ones do follow from the stated ones by permutational symmetry considerations:

$$Z_n \square L^{\otimes n} = Z_n \square R^{\otimes n} \tag{9}$$

$$Z_n \square L^{\otimes(n-1)} \otimes R = Z_n \square R^{\otimes(n-1)} \otimes L \tag{10}$$

$$\vdots$$

$$Z_n \square L \otimes R^{\otimes(n-1)} = Z_n \square R \otimes L^{\otimes(n-1)}. \tag{11}$$

In terms of $\varphi$ and $\psi$, the above system of equations is equivalent to the validity, identically in $(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}$, of the following system of functional equations:

$$\bar{\varphi}(\sum_{k=1}^{n-1} t_k) \prod_{k=1}^{n-1} \varphi(t_k) = \bar{\psi}(\sum_{k=1}^{n-1} t_k) \prod_{k=1}^{n-1} \psi(t_k) \tag{12}$$

$$\bar{\varphi}(\sum_{k=1}^{n-1} t_k) \psi(t_1) \prod_{k=2}^{n-1} \varphi(t_k) = \bar{\psi}(\sum_{k=1}^{n-1} t_k) \varphi(t_1) \prod_{k=1}^{n-1} \psi(t_k) \tag{13}$$

$$\vdots$$

$$\bar{\varphi}(\sum_{k=1}^{n-1} t_k) \prod_{k=1}^{n-1} \psi(t_k) = \bar{\psi}(\sum_{k=1}^{n-1} t_k) \prod_{k=1}^{n-1} \varphi(t_k). \tag{14}$$

STEP 8. *The validity of the system of functional equations* (12)–(14) *is equivalent to condition* $(\widehat{\mathrm{B}}_n)$.

PROOF. Assume that at least one of the two conditions is true. This implies the identity

$(\widehat{\mathrm{B}}_2)$ $\varphi^2 = \psi^2$.

[Condition $(\widehat{\mathrm{B}}_n)$ implies $(\widehat{\mathrm{B}}_2)$ by considering $t \in \mathbb{R}^{n-1}$ with $t_k = 0$ for $k \geq 2$. Similarly, (12) and (13) first imply

$$\bar{\varphi}\varphi = \bar{\psi}\psi \quad \text{and} \quad \bar{\varphi}\psi = \bar{\psi}\varphi. \tag{15}$$

If $t \in \mathbb{R}$ and $\varphi(t)\psi(t) = 0$, then the first equation in (15) immediately yields $\varphi^2(t) = \psi^2(t)$. For the remaining $t$, devide left hand sides and right hand sides, respectively, of the two equations in (15).]

Now $(\widehat{\mathrm{B}}_2)$ implies in particular that each of the $n+1$ equations (12)–(14) and $(\widehat{\mathrm{B}}_n)$ is true whenever the following condition is *not* true:

$$\varphi(\sum_{k=1}^{n-1} t_k)\psi(\sum_{k=1}^{n-1} t_k) \prod_{k=1}^{n-1} \varphi(t_k)\psi(t_k) \neq 0. \tag{16}$$

So assume (16). Then, using ($\widehat{\mathrm{B}}_2$), each of the equations (12)–(14) and ($\widehat{\mathrm{B}}_n$) is seen to be equivalent to the following: The number of the true equations among $\varphi(t_1) = -\psi(t_1)$, $\ldots$, $\varphi(t_{n-1}) = -\psi(t_{n-1})$, $\varphi(\sum_{k=1}^{n-1} t_k) = -\psi(\sum_{k=1}^{n-1} t_k)$ is even.

Thus both of the two conditions (12)–(14) and ($\widehat{\mathrm{B}}_n$) are true. ∎

**3.3 Proof of Theorem 1.2.** We may assume the validity of the conditions ($\widehat{\mathrm{B}}_n$) for $n = 1$ and $n = 2$, which read

($\widehat{\mathrm{B}}_1$) $\varphi(0) = \psi(0)$,

($\widehat{\mathrm{B}}_2$) $\varphi^2 = \psi^2$.

If $\varphi(0) = 0$, then $\psi(0) = 0$ and, using $P \geq 0$, $\varphi = 0$ and $\psi = 0$. Since this trivially yields (S), we may assume $\varphi(0) > 0$ in what follows.

Use continuity of $\varphi$ to choose a $\delta > 0$ with

$$|\varphi(t)| \;>\; 0 \qquad (t \in [-\delta, \delta]). \tag{17}$$

By continuity again, ($\widehat{\mathrm{B}}_1$) and ($\widehat{\mathrm{B}}_2$) yield

$$\varphi(t) \;=\; \psi(t) \qquad (t \in [-\delta, \delta]). \tag{18}$$

a) Assume now that ($\mathrm{B}_n$) is true for every $n \in \mathbb{N}$. Let $T \in \mathbb{R}$. Choose $n \in \mathbb{N}$ and $t \in [-\delta, \delta]$ with $T = (n-1)t$, and apply ($\widehat{\mathrm{B}}_n$) to the vector $(t, \ldots, t) \in \mathbb{R}^{n-1}$. This yields

$$\varphi(T)\varphi^{n-1}(t) \;=\; \psi(T)\psi^{n-1}(t).$$

By (17) and (18), $\varphi(T) = \psi(T)$. Thus ($\widehat{\mathrm{S}}$), hence (S).

b) Now assume ($\mathrm{FM}_+$). [As promised in 2.3, only this weakening of condition (FM) will be used.] In terms of $\mu$ and $\nu$, this reads

$$\int e^{\varepsilon x}\, d\nu(x) < \infty. \tag{19}$$

for some $\varepsilon \in \mathbb{R} \setminus \{0\}$. Applying Theorem 3 of Marcinkiewicz (1938), we see that ($\widehat{\mathrm{S}}$) follows from (18), (19), and nonnegativity of $\mu$ and $\nu$.

Remark. For additional information, apparantly not to be found elsewhere, on the relation between the conditions (18) and ($\widehat{\mathrm{S}}$), see Hsu (1954). That (18) alone is not sufficient for ($\widehat{\mathrm{S}}$) is well known since Gnedenko (1937).

c) Let $n \in \mathbb{N}$. Let $\varphi_0$ be a characteristic function with support contained in $[-1, 1]$. For $t \in \mathbb{R}$, put

$$\begin{aligned}
\varphi(t) &= \frac{1}{2}\left(\varphi_0(t) + \frac{1}{2}\varphi_0(t-n) + \frac{1}{2}\varphi_0(t+n)\right), \\
\psi(t) &= \frac{1}{2}\left(\varphi_0(t) - \frac{1}{2}\varphi_0(t-n) - \frac{1}{2}\varphi_0(t+n)\right).
\end{aligned}$$

Then $\varphi$ and $\psi$ are Fourier transforms of two different nonnegative measures $\mu$ and $\nu$. [If $\varphi_0 = \widehat{\mu}_0$, then $2\varphi(0) = 1$ and $2\varphi(t) = \int e^{itx}(1 + \cos nx)\, d\mu_0(x)$. Thus $2\varphi$ is the Fourier transform of the probability measure with $\mu_0$–density $x \mapsto 1 + \cos nx$. For $\psi$, replace cos by $-$ cos.]

Now define $L$ and $R$ by (4) and (5), and finally define $P \in \mathrm{Prob}(\mathbb{R})$ by (3) with $c = 0$. By construction, $P$ does not satisfy $(\widehat{\mathrm{S}})$.

To check that $P$ satisfies $(\widehat{\mathrm{B}}_n)$, and hence $(\mathrm{B}_n)$, observe first that $P$ satisfies $(\widehat{\mathrm{B}}_2)$. Hence we may restrict attention to those $t \in \mathbb{R}^{n-1}$ satisfying (16) and hence $(\sum t_k, t_1, \ldots, t_{n-1}) \in S^n$, where

$$S \;=\; ]-n-1, -n+1[ \,\cup\, ]-1, 1[ \,\cup\, ]n-1, n+1[.$$

A simple checking of possible cases shows that the condition $(\sum t_k, t_1, \ldots, t_{n-1}) \in S^n$ implies that

$$1(\sum_{k=1}^{n-1} t_k \in S\backslash]-1,1[\,) + \sum_{k=1}^{n-1} 1(\,t_k \in S\backslash]-1,1[\,)$$

is an even number. Since $\varphi(t) = -\psi(t)$ for $t \in S\backslash]-1,1[$, and $\varphi(t) = \psi(t)$ otherwise, the equation in $(\widehat{\mathrm{B}}_n)$ follows. ∎

**3.4 Proof of the claim in Remark 2.3 concerning logarithmic moments.** It is well known that compactly supported characteristic functions exist which are infinitely often differentiable. [Take the convolution square of any real symmetric compactly supported and infinitely often differentiable function, and normalize it to have the value one at zero.] If the $\varphi_0$ in part c) of the above proof is accordingly chosen, then the resulting asymmetric $P$ has logarithmic moments of all orders. ∎

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