Convolution operators on spaces of holomorphic functions

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Abstract

A class of convolution operators on spaces of holomorphic functions related to the Hadamard multiplication theorem for power series and generalizing infinite order Euler differential operators is introduced and investigated. Emphasis is placed on questions concerning injectivity, denseness of range, and surjectivity of the operators.

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1 Introduction

Let $U$ be an open subset of the punctured plane $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$ and let $H(U)$ denote the space of functions holomorphic in $U$ with the usual topology of locally uniform convergence. If $L \subset U$ is compact, a cycle $\Gamma$ in $U \setminus L$ is called a Cauchy cycle for $L$ in $U$ if

$$\text{ind}_\Gamma(w) = 1 \quad (w \in L) \quad \text{and} \quad \text{ind}_\Gamma(w) = 0 \quad (w \in \mathbb{C} \setminus U)$$

(note that we always require $\text{ind}_\Gamma(0) = 0$). According to [19, Theorem 13.5], for each pair $(U, L)$ as above a Cauchy cycle exists. For basic notations and facts concerning cycles we refer to [19, Chapter 10]. If $f \in H(U)$ and if $\Gamma$ is a Cauchy cycle for $L$ in $U$, then Cauchy’s theorem ([19, Theorem 10.35]) implies that

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{1 - z/\zeta} \frac{f(\zeta)}{\zeta} d\zeta \quad (z \in L).$$

Let in the sequel $\Omega$ always be open in $\mathbb{C}_*$ and so that $\Omega \cup \{0, \infty\}$ is open in $\mathbb{C}_\infty$, where $\mathbb{C}_\infty$ denotes the extended plane. In this case we say that $\Omega$ is spherically **University of Trier, FB IV, Mathematics, D-54286 Trier, Germany, e-mail: lors4501@uni-trier.de; jmueller@uni-trier.de**
open. If, in addition, \( \Omega \) is connected, we call \( \Omega \) a spherical domain. Moreover, we consider a function \( \varphi \in H(\Omega) \) with
\[
\varphi(z) = \sum_{\nu=0}^{\infty} \varphi_{\nu}z^{\nu}
\]
near 0 and
\[
\varphi(z) = \sum_{\nu=1}^{\infty} -\varphi_{-\nu}z^{-\nu}
\]
near \( \infty \). Finally, for \( A, B \subset \mathbb{C}_* \) we define \( A^* := 1/(\mathbb{C}_* \setminus A) \) (with \( 1/\emptyset := \emptyset \)) and
\[
A * B := (A^*B^*)^*
\]
where, as usual, \( CD := \{zw : z \in C, w \in D\} \) for \( C, D \subset \mathbb{C} \). The following facts play a basic role: For an arbitrary set \( S \subset \mathbb{C}_* \) we have
\[
S \subset A * B \text{ if and only if } S \cdot A^* \subset B.
\]
Moreover, if \( S \) is compact, then \( S \cdot \Omega^* \) is compact.

Let \( f \in H(U), L \subset \Omega^*U \) compact, and \( \Gamma = \Gamma_L \) a Cauchy cycle for \( L \cdot \Omega^* \) in \( U \). Then \( z/\zeta \in \Omega \) for \( z \in L \) and \( \zeta \in \Gamma \), and from Cauchy’s theorem one can deduce that
\[
(\varphi * f)(z) = (\varphi *_{\Omega,U} f)(z) := \frac{1}{2\pi i} \int_{\Gamma} \varphi\left(\frac{z}{\zeta}\right) \frac{f(\zeta)}{\zeta} d\zeta \quad (z \in L)
\]
(2)
independent of the choice of \( \Gamma \) defines a function \( \varphi * f \in H(\Omega^*U) \). Moreover, the mapping
\[
H(\Omega) \times H(U) \ni (\varphi, f) \mapsto \varphi * f \in H(\Omega^*U)
\]
is bilinear and continuous (cf. [8], [16], [17]).

Let \( V \) be spherically open and \( B \subset V \) closed in \( \mathbb{C}_* \). We call a cycle \( \Gamma \subset V \setminus B \) an anti-Cauchy cycle for \( B \) in \( V \) if and only if \( z/\Gamma \) is a anti-Cauchy cycle for \( z \cdot U^* \) in \( \Omega \) (see [17]).

If \( f \in H(U), L \subset \Omega^*U \) compact, and \( \Gamma \) an anti-Cauchy cycle for \( L \cdot U^* \) in \( \Omega \) we define
\[
(f * \varphi)(z) = (f *_{U,\Omega} \varphi)(z) := \frac{1}{2\pi i} \int_{\Gamma} f\left(\frac{z}{w}\right) \frac{\varphi(w)}{w} dw \quad (z \in L)
\]
(independent of the choice of \( \Gamma \)). Then the substitution \( \zeta = z/w \) leads to
\[
\varphi * f = f * \varphi
\]
that is, the convolution product is commutative.

We emphasize a special case: If $\Omega = C_s \setminus \{1\}$ then $\Omega \ast U = U$ for arbitrary open sets $U$ as above and thus $\varphi \ast f \in H(U)$. In the case of the Cauchy kernel $\varphi = 1_s \in H(C_s \setminus \{1\})$, i.e. $1_s(z) := 1/(1 - z)$, we obtain $1_s \ast f = f$.

The main objective of this paper is the investigation of the (continuous) operator $T_\varphi = T_{\varphi,U} : H(U) \to H(\Omega \ast U)$ defined by

$$T_\varphi(f) := \varphi \ast f \quad (f \in H(U))$$

and of natural restrictions of this operator which we are going to introduce now:

For $\rho > 0$ we define $\tau_\rho(t) := \rho e^{it} \ (t \in [0, 2\pi])$ to be the standard parametrization of the circle $\{|z| = \rho\}$. Moreover, we set $p_\nu(z) := z^\nu$ for $z \in C_s$ and $\nu \in \mathbb{Z}$.

Taking $\Gamma$ as the sum of $\tau_R$ and $\tau_r$ in (2), where $R$ is sufficiently large and $r$ sufficiently small, the above expansions for $\varphi$ lead to the basic property $\varphi \ast p_\nu = \varphi_\nu \cdot p_\nu$ (3) and in particular to $(\varphi \ast p_\nu)(1) = \varphi_\nu$ (note that $\Omega \ast (C_s \setminus \{1\})$).

In the special case $\Omega = C_s \setminus \{1\}$ we have $T_\varphi : H(U) \to H(U)$ and thus (3) shows that the monomials $p_\nu$ are eigenfunctions corresponding to the eigenvalues $\varphi_\nu$. In this case, the operator $T_\varphi$ may be written as an infinite order Euler differential operator (cf. [9, Section 11.2]). Euler differential operators and coefficient multipliers on spaces of real analytic functions were rigorously investigated in a series of publications by Domanski and Langenbruch ([4], [5], [6]).

If $U \cup \{0\}$ is open in $C_s$ and $f(z) = \sum_{\nu=0}^{\infty} f_\nu z^\nu =: f^+(z)$ near 0 (that is, $f$ has a removable singularity at 0), we obtain from (3)

$$\varphi \ast (\varphi \ast f)(z) = \sum_{\nu=0}^{\infty} \varphi_\nu f_\nu z^\nu$$

near 0. This is the Hadamard multiplication theorem in a general form (see e.g. [8], [16], [17], and [9, Theorem 11.6.1], [20, Theorem 3.2] for the classical "star-like" version). Because of this connection, we call $\varphi \ast f$ Hadamard convolution product of $\varphi$ and $f$.

Similarly, if $U \cup \{\infty\}$ is open in $C_\infty$ and $f(z) = \sum_{\nu=1}^{\infty} -f_\nu z^{-\nu} =: f^-(z)$ near $\infty$ then

$$\varphi \ast f(z) = \sum_{\nu=1}^{\infty} -\varphi_\nu f_\nu z^{-\nu}$$

near $\infty$.

We write $H^+(U)$ for the closed subspace of $H(U)$ consisting of those functions having a removable singularity at 0 (in case that $U \cup \{0\}$ is open) and $H^-(U)$ for the closed subspace of functions that vanish at $\infty$ (in case that $U \cup \{\infty\}$ is open). By $H^\pm(U)$ we denote the intersection of both spaces (if $U$ is spherically open). According to our assumptions, we always have $\varphi \in H^\pm(\Omega)$.

If we put

$$M_+ := M \cup \{0\}, \quad M_- := M \cup \{\infty\}, \quad M_\pm := M \cup \{0, \infty\},$$

then...
for \( M \subset \mathbb{C}_* \), then \( H^+(U) \) is in an obvious way isomorphic to \( H(U_+) \), and similarly \( H^-(U) \cong H(U_-) \) and \( H^\pm(U) \cong H(U_\pm) \). With that, we define

\[
T^+_\varphi := T\varphi|_{H^+(U)}, \quad T^-\varphi := T\varphi|_{H^-(U)}, \quad T^\pm\varphi := T\varphi|_{H^\pm(U)}.
\]

According to (4) we have \( T^+_\varphi(U) \subset H^+(\Omega \ast U) \) and, similarly, (5) implies \( T^-\varphi(U) \subset H^-(\Omega \ast U) \) and \( H^\pm(\Omega \ast U) \), respectively.

The operators are already introduced in [17], where, however, the definition of \( T\varphi \) is given for subsets of the extended plane \( \mathbb{C}_\infty \) instead of the punctured plane \( \mathbb{C}_* \). This approach requires the distinction of a number of different cases depending on whether 0 or \( \infty \) belong to \( U \) or not. The above approach reduces the underlying calculations considerably.

The paper is arranged as follows: In the next section we formulate first basic facts about the kernel and the range of the operators. In Section 3, the Mellin transform is used to obtain further results in this direction. In Section 4 we use Koethe duality in order to describe the dual operator of \( T\varphi \). The main ingredient for the proof is an associative law for the Hadamard convolution product. Finally, in the last two sections, applications of duality concerning injectivity, denseness of the range and surjectivity are given.

## 2 kernel and denseness of the range

We start with some basic statements about the kernel and the range of the above convolution operators. We define

\[
N_{\varphi^+} := \{ \nu \in \mathbb{N}_0 : \varphi_\nu = 0 \}, \quad N_{\varphi^-} := \{ \nu \in \mathbb{N} : \varphi_{-\nu} = 0 \}
\]

and \( N_{\varphi} := N_{\varphi^+} \cup (-N_{\varphi^-}) \).

### 2.1 Remark.

Denoting by \( \text{span}A \) the span of a subset \( A \) of a linear space, from (3) it follows immediately that

\[
\text{span}\{ p_\nu : \nu \in N_{\varphi^+} \} \subset \ker(T\varphi)
\]

and

\[
\text{span}\{ p_\nu : \nu \in \mathbb{Z} \setminus N_{\varphi} \} \subset \text{im}(T\varphi)
\]

(with \( \text{span}\emptyset = \{0\} \)). In particular, we see that \( N_{\varphi} = \emptyset \) is necessary for the injectivity of \( T\varphi \). If \( \Omega \ast U \) is a ring domain (that is, the complement with respect to \( \mathbb{C}_\infty \) has two components, one containing 0 and one \( \infty \)), then Runge’s approximation theorem shows that \( \text{span}\{ p_\nu : \nu \in \mathbb{Z} \} \) is dense in \( H(\Omega \ast U) \) and thus \( N_{\varphi} = \emptyset \) is sufficient for \( T\varphi \) to have dense range.

If \( U \) is a ring domain of the form \( U = \{ r < |z| < R \} \), for some \( 0 \leq r < R \leq \infty \), then the Laurent expansion of \( f \in H(U) \) shows that \( \text{span}\{ p_\nu : \nu \in N_{\varphi} \} \) is dense in \( \ker(T\varphi) \). So, in this case, \( T\varphi \) is injective if and only if \( N_{\varphi} = \emptyset \). Finally, if \( \Omega \ast U \)
is a ring domain of the above form, then, again according to Laurent expansion, it is seen that for each $n \in \mathbb{N}$ the monomial $p_n$ does not belong to the closure of $\text{im}(T_\varphi)$ in $H(\Omega*U)$. Thus, in this case, $T_\varphi$ has dense range if and only if $N_\varphi = \emptyset$.

One main purpose of this paper is the investigation of the corresponding questions for other open sets $U$, in particular for $U$ having simply connected components (that is, for $U$ having connected complement with respect to $C_\infty$).

2.2 Remark. To obtain a description of the kernel for the one-sided cases $T_\varphi^+$ and $T_\varphi^-$, we put

$$H_N(U) := \{ f \in H^+(U) : f^+(z) = \sum_{\nu \in N} f_\nu z^\nu \}$$

for $N \subseteq \mathbb{N}_0$ and

$$H_{-N}(U) := \{ f \in H^-(U) : f^-(z) = \sum_{\nu \in N} -f_{-\nu} z^{-\nu} \}$$

for $N \subseteq \mathbb{N}$. Then the expansions (4) and (5), respectively, show that

- $\text{span}\{p_\nu : \nu \in N_\varphi^+\} \subseteq \ker(T_\varphi^+) \subset H_{N_\varphi^+}(U)$ and if $\Omega*U$ is connected, then $\ker(T_\varphi^+) = H_{N_\varphi^+}(U)$.

- $\text{span}\{p_\nu : \nu \in -N_\varphi^-\} \subseteq \ker(T_\varphi^-) \subset H_{-N_\varphi^-}(U)$ and if $\Omega*U$ is connected, then $\ker(T_\varphi^-) = H_{-N_\varphi^-}(U)$.

As a consequence of the first parts we obtain in the case that $U$ is a domain:

- $T_\varphi^+$ is injective if and only if $N_\varphi^+ = \emptyset$.

- $T_\varphi^-$ is injective if and only if $N_\varphi^- = \emptyset$.

2.3 Remark. From (3) and (4) it follows immediately that $N_\varphi^+ = \emptyset$ is necessary for $T_\varphi^+$ to have dense range. Similarly, $N_\varphi^- = \emptyset$ is a necessary condition for $T_\varphi^-$ to have dense range. As an application of Runge’s approximation theorem we get

- If $(\Omega*U)_+$ has simply connected components, then $T_\varphi^+$ has dense range if and only if $N_\varphi^+ = \emptyset$.

- If $(\Omega*U)_-$ has simply connected components, then $T_\varphi^-$ has dense range if and only if $N_\varphi^- = \emptyset$.

The results show that the questions concerning injectivity and denseness of the range are easy to answer in the one-sided cases $T_\varphi^+$ and $T_\varphi^-$ (at least under certain natural conditions on $U$ and $\Omega*U$, respectively). In the sequel we will concentrate on the cases $T_\varphi$ and $T_\varphi^\pm$.

In a first step we consider $U$ with simply connected components and $\Omega$ of a special form, which leads to essentially more eigenfunctions for $T_\varphi$ than given by (3).
3 Mellin transform and applications

Let \( K \subset \mathbb{S} := \{ w \in \mathbb{C} : |\text{Im}(w)| < \pi \} \) be compact and connected. For \( \varphi \in H((e^K)^*) \), the Mellin transform \( M\varphi \) of \( \varphi \) is given by

\[
M\varphi(\alpha) := \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta^{\alpha+1}} \, d\zeta \quad (\alpha \in \mathbb{C}),
\]

where \( \Gamma \) is a Cauchy cycle for \( e^{-K} \) in \( \mathbb{C} := \mathbb{C} \setminus (-\infty, 0] \), and \( \zeta^c := \exp(c \log \zeta) \) with the principal branch of the logarithm. The (linear) map \( M : H((e^K)^*) \to \text{Exp}(\text{conv}(K)) \) is called Mellin transformation. Here, for \( L \) convex in \( \mathbb{C} \), the space \( \text{Exp}(L) \) is defined as the set of entire functions of exponential type having conjugate indicator diagram contained in \( L \). Endowed with a natural metric, \( \text{Exp}(L) \) turns out to be a Fréchet space (see, e.g. [2, p. 82, pp. 266]).

**3.1 Remark.** 1. The Mellin transformation \( M : H((e^K)^*) \to \text{Exp}(\text{conv}(K)) \) is injective and for \( \Phi := M\varphi \) we have

\[
\varphi_\nu = \Phi(\nu) \quad (\nu \in \mathbb{Z}).
\]

Furthermore, if \( K \) is convex, then \( M \) is also surjective.

The situation in the special case \( K = \{ 0 \} \) is known as the Wigert-Leau Theorem. In this case \( \Omega = \mathbb{C}_+ \setminus \{ 1 \} \) and \( \Phi = M\varphi \) is of exponential type zero. Moreover, we then have \( \Omega * U = U \) and \( T_\varphi \) is an Euler differential operator of the form

\[
T_\varphi f = \Phi(\vartheta)f := \sum_{k=0}^{\infty} \Phi_k \vartheta^k f \quad (f \in H(U))
\]

where \( (\vartheta f)(z) := z f'(z) \) and \( \Phi(\alpha) = \sum_{k=0}^{\infty} \Phi_k \alpha^k \) (see, e.g. [2, pp. 71, pp. 419], [9, Section 11.2], [17]). Conversely, since \( M \) is bijective, for each \( \Phi \) of exponential type zero the operator \( \Phi(\vartheta) \) is of the form \( T_{\Phi^0} \). For \( \Phi(\alpha) = \alpha \) we get the Koebe function \( \kappa(z) := z/(1-z)^2 = \sum_{\nu=0}^{\infty} \nu z^\nu \) and \( T_\varphi f = \vartheta f \).

2. By differentiation of the defining parameter integral one can verify that the derivatives of \( \Phi \) are given by

\[
\Phi^{(k)}(\alpha) := -\frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta^{\alpha+1}} (-1)^k (\log \zeta)^k \, d\zeta \quad (\alpha \in \mathbb{C}, \ k \in \mathbb{N}_0).
\]

For an open set \( U \subset \mathbb{C}_+ \) with simply connected components let

\[
p_\alpha(z) := p_{\alpha, U}(z) := z^\alpha := e^{\alpha \log_U z}
\]

for \( z \in U \) and \( \alpha \in \mathbb{C} \), where \( \log_U \) is defined as a fixed branch of the logarithm on each component of \( U \). More generally, we define

\[
q_{k, \alpha, U} := (\log_U)^k \cdot p_{\alpha, U}
\]

for \( k \in \mathbb{N}_0 \).

In the sequel we frequently use the fact that \( \Omega * U \) has simply connected components if \( U \) has simply connected components (this follows from \( (\Omega * U)^* = \Omega^* U^* = \bigcup_{w \in \Omega^*} w \cdot U^* \), the connectedness of \( \mathbb{C}_\infty \setminus w \cdot U^* \), and \( 0 \in \mathbb{C}_\infty \setminus w \cdot U^* \) for \( w \neq 0 \)). Moreover, we write \( U_\delta(z_0) := \{ z : |z - z_0| < \delta \} \) for \( \delta > 0 \) and \( z_0 \in \mathbb{C} \).
3.2 Theorem. Let $U \subset \mathbb{C}_*$ be open with simply connected components and $\Omega = (e^K)^*$. Then for each logarithm $\log_U$ on $U$ there exists a logarithm $\log_{\Omega \ast U}$ on $\Omega \ast U$ such that

$$T_\varphi q_{k, \alpha, U} = \varphi \ast q_{k, \alpha, U} = p_{\alpha, \Omega \ast U} \sum_{\ell=0}^{k} \binom{k}{\ell} (\log_{\Omega \ast U})^{k-\ell} \Phi(\ell)(\alpha)$$

for all $k \in \mathbb{N}_0$ and in particular

$$T_\varphi p_{\alpha, U} = \varphi \ast p_{\alpha, U} = \Phi(\alpha) \cdot p_{\alpha, \Omega \ast U}.$$

Proof. We fix a branch of the logarithm on each component of $U$ and denote the resulting holomorphic function on $U$ by $\log_U$. Our aim is to show how branches of the logarithm on the components of $\Omega \ast U$ can be chosen in such a way that the asserted identity holds.

It is clear that there exists a number $a \in K$ such that the set $K - a$ contains the origin (if $K$ itself contains the origin we choose $a = 0$). This implies that $1 \notin e^a \cdot \Omega$ and therefore $e^a \cdot (\Omega \ast U) \subset U$. Especially, every component of $e^a \cdot (\Omega \ast U)$ is a subset of a component of $U$. Therefore it is meaningful to set

$$\log_{e^a \cdot (\Omega \ast U)} := \log_U(e^a z) \ (z \in \Omega \ast U).$$

Obviously, every branch of the logarithm on $\Omega \ast U$ fulfills the following equation for all $z \in \Omega \ast U$:

$$\log_{\Omega \ast U}(z) = \log_U(e^a z) - a + 2\pi i k(z)$$

for some $k(z) \in \mathbb{Z}$. The map

$$\Omega \ast U \ni z \mapsto \log_{\Omega \ast U}(z) - (\log_U(e^a z) - a) \in \mathbb{C}$$

is continuous and its range is a discrete subset of $\mathbb{C}$. Therefore it must be constant on every component of $\Omega \ast U$. Hence, the branch of the logarithm on every component of $\Omega \ast U$ shall be chosen such that

$$\log_{\Omega \ast U}(z) = \log_U(e^a z) - a \ (z \in \Omega \ast U). \quad (7)$$

Let now $z \in \Omega \ast U$ be given. The set $z \cdot U^*$ is a compact subset of the open set $\Omega$ (see (1)) and therefore we can find a number $\delta_1 = \delta_1(z) > 0$ such that $(e^{-K} + U_{\delta_1}(0)) \cap z \cdot U^* = \emptyset$. On the other hand, $e^{-K}$ is a compact subset of the open set $\mathbb{C}_-$ and therefore we can find a number $\delta_2 > 0$ such that $e^{-K} + U_{\delta_2}(0) \subset \mathbb{C}_-$. We set $\delta := \min\{\delta_1, \delta_2\}$ and $V_\delta := e^{-K} + U_{\delta}(0)$.

The following functional equation is essential for the rest of the proof: For all $z \in V_\delta$ we have

$$\log_U(\frac{z}{\zeta}) = \log_{\Omega \ast U}(z) - \log(\zeta) \quad (8)$$

where $\log$ denotes the principal branch of the logarithm on $\mathbb{C}_-$. In order to prove (8) we first of all note that the left-hand side of (8) is defined since $z/\zeta \in U$ for all $\zeta \in V_\delta$. Indeed, assuming the existence of a number
$w \in U^C$ with $z/\zeta = w$ would imply $z \cdot U^* \ni z/w = \zeta \in V_{\delta}$ which contradicts the choice of $\delta$.

Obviously we have

$$g_z(\zeta) := \log_U(z\zeta) - (\log_{\Omega^*}(z) - \log(\zeta)) = 2k_z(\zeta)\pi i \quad (\zeta \in V_{\delta})$$

for some $k_z(\zeta) \in \mathbb{Z}$. The same argument as above yields that $g_z$ is constant on $V_{\delta}$ (note that $V_{\delta}$ is connected). Inserting $\zeta_0 = e^{-a} \in V_{\delta}$ implies (with (7) and noting that $a \in K$ and therefore $\log(e^{-a}) = -a$)

$$g_z(\zeta) = g_z(\zeta_0) = \log_U(e^a z) - (\log_U(e^a z) - a - \log(e^{-a})) = 0 \quad (\zeta \in V_{\delta}).$$

This completes the proof of the asserted functional equation.

Since $e^{-K}$ is a compact subset of the open set $V_{\delta}$ there exists a Cauchy cycle $\tilde{\Gamma}$ for $e^{-K}$ in $V_{\delta}$. The choice of $\delta$ ensures that

1. $\tilde{\Gamma}$ is a Cauchy cycle for $e^{-K}$ in $\mathbb{C}_-$,

2. $\Gamma := \tilde{\Gamma}^{-}$ is an anti-Cauchy cycle for $z \cdot U^*$ in $\Omega$.

Finally we obtain with Equation (8)

$$T_\varphi q_{k,\alpha,U}(z) = q_{k,\alpha,U} \ast \varphi(z)$$

$$= \frac{1}{2\pi i} \int_\Gamma \varphi(\zeta) \left( \log_U(z\zeta) \right)^k \exp(\alpha \log_U(z\zeta)) \frac{d\zeta}{\zeta}$$

$$= p_{\alpha,\Omega^*U}(z) \sum_{l=0}^{k} \frac{k^l}{l!} \frac{(-1)^{l+1}}{2\pi i} \int_{\tilde{\Gamma}} \varphi(\zeta) (\log \zeta)^l \zeta^{\alpha+1} d\zeta$$

$$= p_{\alpha,\Omega^*U}(z) \sum_{l=0}^{k} \frac{k^l}{l!} \Phi^{(l)}(\alpha).$$

(9)

(10)

**3.3 Corollary.** If $Z(\Phi) := \{ \alpha : \Phi(\alpha) = 0 \}$ then

$$\text{span}\{q_{k,\alpha,U} : \alpha \text{ m-fold zero of } \Phi, \ k \leq m - 1\} \subset \ker(T_\varphi).$$

As a consequence we obtain

**3.4 Theorem.** If $U \subset \mathbb{C}_*$ is a simply connected domain and if $\Omega = (e^K)^*$, then the following are equivalent:

1. $T_\varphi$ is injective,

2. $\Phi$ has no zeros,

3. $\varphi$ is a nonzero multiple of $1_*(e^\beta \cdot)$ for some $\beta \in K$. 

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Proof. Corollary 3.3 yields that 1. implies 2. 
In order to show that 2. implies 3. we assume that \( \Phi \in \text{Exp}(K) \) has no zeros. 
Then according to the Hadamard factorization theorem (see e.g. [3, Th. 2.7.1]), there are numbers \( \alpha, \beta \in \mathbb{C} \) such that 
\( \Phi(z) = \exp(\beta z + \alpha) \) \((z \in \mathbb{C})\). In order that the condition \( \Phi \in \text{Exp}(K) \) is satisfied, \( \beta \) must belong to the set \( K \). 
Setting \( \lambda := e^\alpha \neq 0 \), the power series expansion of \( \varphi \) about zero yields that
\[
\varphi(z) = \frac{\lambda}{1 - e^{\beta z}} \quad (z \in \Omega).
\]

To prove that 3. implies 1. we examine how \( T_\varphi \) acts on a function \( f \in H(U) \).
For all \( z \in \Omega \ast U \) we obtain
\[
T_\varphi f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\lambda}{1 - e^{\beta z}/\zeta} f(\zeta) \frac{d\zeta}{\zeta} = \lambda f(e^{\beta z})
\]
where \( \Gamma \) is a Cauchy cycle for \( z \cdot e^K \) in \( U \) and the last identity follows from the Cauchy integral formula. Since \( U \) is connected, the operator \( T_\varphi \) is injective. \( \square \)

As a direct application of Theorem 3.2 and Runge’s theorem, we get

3.5 Theorem. let \( U \subset \mathbb{C} \ast \) be open with simply connected components and \( \Omega = (e^K) \ast \). If \( \varphi \neq 0 \) then \( T_\varphi \) has dense range.

Proof. Again we write \( \Phi = M \varphi \in \text{Exp}(K) \). Then we obtain with Theorem 3.2
\[
T_\varphi p_{\alpha,U} = \Phi(\alpha) \cdot p_{\alpha,\Omega \ast U} \quad (\alpha \in \mathbb{C}).^1
\]

Denoting the set of zeros of \( \Phi \) by \( Z(\Phi) \), this identity implies that for all \( \alpha \notin Z(\Phi) \) the monomials \( p_{\alpha,\Omega \ast U} \) belong to the range of \( T_\varphi \). Since \( Z(\Phi) \) is countable, there exists a number \( c \in [0,1) \) such that \( (\mathbb{N}_0 + c) \cap Z(\Phi) = \emptyset \).
Let now \( g \in H(\Omega \ast U) \). Then the function \( p_{-c,\Omega \ast U} \cdot g \) is also holomorphic in \( \Omega \ast U \). Since \( \Omega \ast U \) has simply connected components, Runge’s approximation theorem implies that there exists a sequence of polynomials \( (P_n)_{n \in \mathbb{N}} \) converging locally uniformly to \( p_{-c,\Omega \ast U} \cdot g \) on \( \Omega \ast U \). Hence, \( (p_{c,\Omega \ast U} \cdot P_n)_{n \in \mathbb{N}} \) converges locally uniformly on \( \Omega \ast U \) to the function \( g \). This shows that the linear span of \( \{p_{\alpha,\Omega \ast U} : \alpha \in \mathbb{C} \setminus Z(\Phi)\} \) is dense in \( H(\Omega \ast U) \) and therefore \( T_\varphi : H(U) \rightarrow H(\Omega \ast U) \) has dense range. \( \square \)

3.6 Example. We consider the simple but illustrating example \( \Omega = \mathbb{C} \ast \{\pm i\} \),
\[
\varphi(z) = \frac{1}{1 + z^2} = \frac{1}{2} \left( \frac{1}{1 - e^{i\pi/2} z} + \frac{1}{1 - e^{-i\pi/2} z} \right) \quad (z \in \Omega).
\]
If \( U = \mathbb{C} \) then \( \Omega \ast U = \{z : \text{Re}(z) \neq 0\} \). Since \( \varphi \) is even, by definition the same holds for \( \varphi \ast f \) for all \( f \in H(U) \) (take \( L = \{\pm z\} \) in (2)). Thus, \( \text{im}(T_\varphi) \) is not

\( ^1 \)The functions \( p_{\alpha,U} \) are assumed to be induced by an arbitrary branch of the logarithm on each component of \( U \).
dense in $H(\Omega \ast U)$. On the other hand, since $M_{\alpha}(\exp)(\alpha) = e^{\beta \alpha}$ for $\alpha, \beta \in \mathbb{C}$, we see that

$$\Phi(\alpha) = M_{\varphi}(\alpha) = \cos(\alpha \pi / 2) \quad (\alpha \in \mathbb{C}).$$

The (conjugate) indicator diagram of $\Phi$ is the line segment $K = i[-\pi/2, \pi/2]$ on the imaginary axis. Therefore, $(e^K)^* \ast U = \{z : \text{Re}(z) > 0\}$ is the right half plane. According to Theorem 3.5, $\text{im}(T_{\varphi}(e^K)^*)$ is dense in $H((e^K)^* \ast U)$.

For the special case $K = \{0\}$ we obtain from Theorems 3.4 and 3.5

3.7 Corollary. Let $U \subset \mathbb{C}_*$ be open with simply connected components and let $\Phi \neq 0$ be an entire function of exponential type 0. Then

1. $\Phi(\vartheta)$ has dense range.
2. $\Phi(\vartheta)$ is injective if and only if $\Phi(\vartheta)$ is a nonzero multiple of the identity on $H(U)$.

3.8 Remark. If $U(= \Omega \ast U)$ is a ring domain of the form $\{r < |z| < R\}$ and if $\Phi$ has a zero at some integer, then $\Phi(\vartheta) : H(U) \rightarrow H(U)$ has no dense range (see Remark 2.1). For instance, if $\Phi(\alpha) = \alpha$, that is, $T_{\varphi}f(z) = \vartheta f(z) = zf'(z)$, then the function $g = 1$ does not belong to the closure of im($T_{\varphi}$).

4 Transpose of $T_{\varphi}$

In order to describe the transpose of $T_{\varphi}$ it is important to provide an associative law for the Hadamard convolution product.

Let $U \subset \mathbb{C}_*$ be open and $K \subset U$ compact. The hull $h_U(K)$ of $K$ with respect to $U$ is defined as the union of $K$ and all relatively compact components of $U \setminus K$. For a cycle $\Gamma$ we denote by $|\Gamma|$ the trace of $\Gamma$, i.e. the union of the images of the closed paths constituting $\Gamma$.

4.1 Theorem. Let $U \subset \mathbb{C}_*$ be open and $\Omega, V \subset \mathbb{C}_*$ spherically open. If $f \in H(U), g \in H^\pm(V)$, and $\varphi \in H^\pm(\Omega)$ then

$$g \ast (\varphi \ast f) = (g \ast \varphi) \ast f.$$

Proof. Let $w \in V \ast (\Omega \ast U)^*$. We choose $\Gamma_1$ to be an anti-Cauchy cycle for $w \cdot (\Omega \ast U)^*$ in $V$. Then

$$\text{ind}_{\Gamma_1}(z) = 0 \quad (z \in w \cdot (\Omega \ast U)^*), \quad \text{ind}_{\Gamma_1}(z) = -1 \quad (z \in V^C) \quad (11)$$

and we obtain additionally

$$\text{ind}_{\Gamma_1}(z) = 0 \quad (z \in (h_{(\Omega \ast U)^*/w}(1/|\Gamma_1|))^*). \quad (12)$$

Indeed, by definition of the hull, it is clear that $|\Gamma_1| \cap (h_{(\Omega \ast U)^*/w}(1/|\Gamma_1|))^* = \emptyset$ and that each component of $(h_{(\Omega \ast U)^*/w}(1/|\Gamma_1|))^*$ meets a component of $w \cdot (\Omega \ast U)^*$. Therefore (12) is a direct consequence of (11). We obtain

$$(g \ast (\varphi \ast f))(w) = ((\varphi \ast f) \ast g)(w) = \frac{1}{2\pi i} \int_{\Gamma_1} (\varphi \ast f)(\frac{w}{t})g(t)\frac{dt}{t}.\quad (10)$$
Now we choose $\Gamma_2$ to be a Cauchy cycle for $(w/|\Gamma_1|) \cdot \Omega^*$ in $U$. Actually, we impose a stronger condition and require $\Gamma_2$ to be a Cauchy cycle for $w \cdot h_{(\Omega^*U)/w}(1/|\Gamma_1|) \cdot \Omega^*$ in $U$. This is possible since the choice of $\Gamma_1$ ensures that $|\Gamma_1| \cap w \cdot (\Omega^*U)^* = \emptyset$. Hence, $(1/|\Gamma_1|)$ and consequently $h_{(\Omega^*U)/w}(1/|\Gamma_1|)$ is a compact subset of $(\Omega^*U)/w$. This, in turn, implies that $w \cdot h_{(\Omega^*U)/w}(1/|\Gamma_1|) \cdot \Omega^*$ is a compact subset of $U$ (see (1)).

We remark that the index property for $\Gamma_1$ implies

$$V^* \subset h_{(\Omega^*U)/w}(1/|\Gamma_1|)$$

and therefore

$$(w \cdot (1/|\Gamma_1|) \cdot \Omega^*) \cup (w \cdot V^* \cdot \Omega^*) \subset w \cdot h_{(\Omega^*U)/w}(1/|\Gamma_1|) \cdot \Omega^*. \quad (13)$$

This yields

$$\left( \varphi \ast f \right)(\frac{w}{t}) = \frac{1}{2\pi i} \int_{\Gamma_2} \varphi \left( \frac{w}{t} \zeta \right) f(\zeta) \frac{d\zeta}{\zeta} \quad (t \in \Gamma_1)$$

and thus

$$\left( g \ast (\varphi \ast f) \right)(w) = \frac{1}{2\pi i} \int_{\Gamma_1} g(t) \frac{1}{t} \frac{1}{2\pi i} \int_{\Gamma_2} \varphi \left( \frac{w}{t} \zeta \right) f(\zeta) \frac{d\zeta}{\zeta} dt$$

$$= \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(\zeta)}{\zeta} \frac{1}{2\pi i} \int_{\Gamma_1} \varphi \left( \frac{w}{t} \zeta \right) g(t) \frac{dt}{\zeta} d\zeta.$$
2. $|\Gamma_2| \cap w \cdot (\Omega \ast V)^* = |\Gamma_2| \cap w \cdot \Omega^* \cdot V^* = \emptyset$ due to (13) and the choice of $\Gamma_2$.

3. A second consequence of (13) and the choice of $\Gamma_2$ is

$$\text{ind}_{\Gamma_2}(z) = 1 \quad (z \in w \cdot \Omega^* \cdot V^*),$$
$$\text{ind}_{\Gamma_2}(z) = 0 \quad (z \in U^C).$$

Hence, $\Gamma_2$ is as a Cauchy cycle for $w \cdot (\Omega \ast V)^*$ in $U$. Therefore, we obtain

$$\frac{1}{2\pi i} \int_{\Gamma_2} (\varphi \ast g)(\frac{w}{\zeta}) f(\zeta) \frac{d\zeta}{\zeta} = ((\varphi \ast g) \ast f)(w) = ((g \ast \varphi) \ast f)(w).$$

For a closed set $B \subset \mathbb{C}_\infty$ we define

$$H(B) := \{[(g,D)]_B : D \supset B \text{ open}, g \in H(D)\}$$

(germs of holomorphic functions on $B$) with the inductive topology corresponding to the restriction maps

$$(j_D : H^\infty(D) \to H(B))_{D \supset B \text{ open}}$$

(see, e.g. [15, p. 292]). If $B \subset \mathbb{C}_*$ is closed, then also $B_{\pm} \subset \mathbb{C}_\infty$ is closed. By identifying $H(D_{\pm})$ and $H^\pm(D)$ for $D$ spherically open, we get

$$H(B_{\pm}) = \{[(g,D)]_{B_{\pm}} : D \supset B \text{ spherically open}, g \in H^\pm(D)\}.$$  

4.2 Remark. (Köthe duality) With the above notations, the well known representation of the dual space $H(U)'$ given by Köthe can be formulated in terms of the convolution product (cf. [8]):

Let $U \subset \mathbb{C}_*$ be open. Then to each $u \in H(U)'$ there corresponds a unique germ $[(g,D)]_{(U^*)}\pm \in H((U^*)\pm)$ such that

$$u(f) = (g \ast f)(1) \quad (f \in H(U)).$$

In the sequel we identify $u$ and $[(g,D)]_{(U^*)}\pm$ and write also $g$ for short.

Note that $(\Omega \ast U)^* = \Omega^* U^*$ for the following.

4.3 Theorem. Let $U \subset \mathbb{C}_*$ open. Then $T_\varphi : H((\Omega^* U^*)\pm) \to H((U^*)\pm)$ is given by

$$T_\varphi[(g,V)]_{(\Omega^* U^*)}\pm = [(g \ast \varphi, V \ast \Omega)]_{(U^*)}\pm$$

or, briefly, $T_\varphi g = g \ast \varphi (= \varphi \ast g)$.

Proof. For an open superset $D$ of $(\Omega^* U)^*$ we have $D^* \subset \Omega^* U$, and (1) yields $(\Omega^* D)^* = \Omega^* \cdot D^* \subset U$ and therefore $\Omega \ast D \supset U^*$. Hence $[(\varphi \ast g, \Omega \ast V)]_{(U^*)}\pm$ belongs to the space $H((U^*)\pm)$. Moreover, $[(g \ast \varphi, V \ast \Omega)]_{(U^*)}\pm$ is independent of the choice of the representative $(g,V)$. 

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We have to check that the unique germ corresponding to $T'_\varphi g \in H(U)'$ is given by $[(g * \varphi, V * \Omega)]_{(U*)_k} \in H((U*)_k)$. We apply the associative law for the Hadamard convolution product and obtain, for all $f \in H(U)$,

$$T'_\varphi g(f) = g(T_\varphi f) = (g * (\varphi * f))(1) = ((g * \varphi) * f)(1) = (g * \varphi)(f).$$

4.4 Remark. According to the Hahn-Banach theorem, for appropriate $U$ as in the introduction we have (in the sense of Kőthe duality)

$$H^+(U)' = H((U^*)_+), \quad H^-(U)' = H((U^*)_-), \quad H^\pm(U)' = H(U^*)$$

and from that it follows that $T^+_\varphi : H((\Omega^*U^*)_+) \to H((U^*)_+)$ is given by

$$(T^+_\varphi)[(g, V)]_{(\Omega^*U^*)_+} = [(g * \varphi, V * \Omega)]_{(U^*)_+}$$

and similarly for $(T^-_\varphi)' : H((\Omega^*U^*)_-) \to H((U^*)_-)$ and $(T^\pm_\varphi)' : H(\Omega^*U^*) \to H(U^*)$.

5 Applications of the Transpose

In this section we will derive further results concerning injectivity and denseness of the range via duality. We use the fact that a continuous linear operator between locally convex spaces has dense range if and only if the transpose is injective.

As a dual version of Theorems 3.4 and 3.5 we obtain

5.1 Theorem. Let $\Omega = (e^K)^*$ and suppose that $\Omega * U$ is a spherical domain.

1. If $U$ is a domain and if $\varphi \neq 0$ then $T^\pm_\varphi$ is injective,

2. $T^\pm_\varphi$ has dense range if and only if $K = \{\beta\}$ and $\varphi$ is a nonzero multiple of $1_*(e^\beta \cdot)$ for some $\beta \in K$.

Proof.

1. Note that since $\Omega * U$ is a spherical domain there exists a closed and connected set $L \subset \Omega * U$ with $0, \infty$ belonging to the interior of $L_\pm \subset \mathcal{C}_\infty$. We set $W := L^*$ and obtain an open set in $\mathcal{C}_*$ having connected complement. Furthermore, we have $W^* = L \subset \Omega * U$ and (1) yields that $(\Omega * W)^* = W^* \cdot \Omega^*$ is a compact subset of $U$.

Theorem 3.5 shows that the operator $T_{\varphi, W} : H(W) \to H(\Omega * W)$ has dense range. Hence

$$T^d_{\varphi, W} : H((\Omega^*W^*)_\pm) \to H((W^*)_\pm)$$

is injective.

Let now $f \in \ker(T^\pm_\varphi)$ be given. Then $[(f, U)]_{(\Omega^*W^*)_k} \in H((\Omega^*W^*)_k)$ and

$$T^d_{\varphi, W}[(f, U)]_{(\Omega^*W^*)_k} = [(\varphi * f, \Omega * U)]_{(W^*)_k} = [0]_{(W^*)_k}.$$
Hence, \([f,U]|_{\Omega^*W^*}\) which means that \(f\) vanishes in an open neighbourhood \(O\) of \(\Omega^*W^*\). Since \(O \cup U \neq \emptyset\) and since \(U\) is connected, \(f\) vanishes on \(U\).

2. If \(K = \{\beta\}\) and \(\varphi = \lambda \mathbf{1}_* (e^{\beta} \cdot )\) then \(T^\pm_\varphi f = \lambda f (e^{\beta} \cdot )\) (see the proof of Theorem 3.4). In this case, \(T^\pm_\varphi : H^\pm(U) \to H^\pm(e^{-\beta}U)\) obviously has dense range (actually, \(T^\varphi\) is surjective).

Conversely, we suppose that \(T^\pm_\varphi\) has dense range. Then \((T^\pm_\varphi)' : H(\Omega^*U^*) \to H(U^*)\) is injective. We can choose \(L\) as in part 1. of the proof so large that \(L\) has no holes lying in \(\Omega \ast U\). Then \(W = L^\ast\) is so that each component contains a point of \(\Omega^*U^*\). This implies that also \(T^\varphi_\ast W : H(W) \to (\Omega \ast W)\) is injective. Theorem 3.4 shows that \(\varphi\) has the desired form. Moreover, then \(K\) has to be a single point set since otherwise \(T^\varphi\) cannot have dense range. \(\Box\)

5.2 Corollary. Let \(U \subset \mathbb{C}_+\) be a spherical domain and let \(\Phi \neq 0\). Then

1. \(\Phi(\vartheta) : H^\pm(U) \to H^\pm(U)\) injective.

2. \(\Phi(\vartheta) : H^\pm(U) \to H^\pm(U)\) has dense range if and only if it is a nonzero multiple of the identity on \(H^\pm(U)\).

5.3 Remark. Note that in the case of a spherical domain \(U\), no monomial \(p_\nu\) belongs to \(H^\pm(U)\). The situation changes drastically if \(U\) has two components \(V\) and \(W\) with \(0 \in V_+\) and \(\infty \in W_-\). In this case, \(p_\nu|_V \in H^+(V)\) for \(\nu \in \mathbb{N}_0\) and \(p_\nu|_W \in H^-(W)\) for \(\nu \in -\mathbb{N}\). If \(\Phi\) has a zero at some integer, then \(\Phi(\vartheta) : H^\pm(U) \to H^\pm(U)\) is no longer injective (according to Remark 2.2, at least one of the operators \(T^\beta_\varphi V\) and \(T^\beta_\varphi W\) is not injective).

We want to formulate a second condition under which the operators \(T^\varphi\) are injective. So far we only considered the case that \(\Omega\) is of the form \((e^K)^\ast\) for connected \(K\), which implies that the Mellin transform of \(\varphi\) exists. Our aim now is to obtain a result for more general spherically open \(\Omega\). It turns out that in some sense non-existence of the Mellin transform may be compensated by imposing conditions on the number of non-vanishing coefficients \(\varphi_\nu\).

For \(X \subset [0, \infty)\) without finite accumulation point let \(n(r) = n_X(r)\) be the number of \(x \in X\) with \(x \leq r\), where \(r > 0\). According to [18, p. 559] (see also [10, p. 178]), the limit

\[
d^\ast(X) := \lim_{\xi \to 1^-} \limsup_{r \to \infty} \frac{n(r) - n(r\xi)}{r - r\xi}
\]

exists and is called the maximal density of \(X\). If the density \(d(X)\) of \(X\), i.e. \(d(X) := \lim_{r \to \infty} n(r)/r\) exists, then \(d(X) = d^\ast(X)\). Moreover, \(d^\ast(X) = 0\) implies the existence of \(d(X)\) (and \(d(X) = 0\)).

In the sequel we use the abbreviations \(K_\delta := i[-\pi \delta, \pi \delta]\) and \(B_\delta := e^{K_\delta}\) for \(\delta \geq 0\). Finally, we write \(\arg z := \text{Im} \log z \in (-\pi, \pi]\).
5.4 Example. Let $N = \{\mu_n : n \in \mathbb{N}\} \subset \mathbb{N}$ with $d(N) = \delta < 1$ and

$$\Phi(\alpha) := \prod_{n=1}^{\infty} \left(1 - \frac{\alpha^2}{\mu_n^2}\right) \quad (\alpha \in \mathbb{C}).$$

Then $\Phi$ is of exponential type with (conjugate) indicator diagram $K_\delta$ (see, e.g., [13, p. 205]) and $Z(\Phi) = \pm N$. Hence $\varphi := M^{-1}\Phi$ is holomorphic in $\Omega = B_\delta^*$ and $N_{\varphi^+} = N$.

1. If $\bar{U} = \mathbb{C}$ then $\Omega \ast U = \{z : |\arg(z)| < \pi(1 - \delta)\}$ and Theorem 3.5 yields that the operator $T_{\varphi}$ has dense range. Example 3.6 is embedded as the special case $N = 2n_0 + 1$.

2. If $U = B_{\eta}^*$ for some $\eta < 1 - \delta$, then $\Omega \ast U = B_{\delta + \eta}^*$. According to Theorem 5.1, $T_{\varphi}^\pm$ is injective.

We show more generally

5.5 Theorem. Let $U$ be a spherical domain and suppose that $B_\delta^* \ast W$ is a domain for some spherically open set $W \subset U$ and some $0 \leq \delta < 1$.

1. If $N \subset \mathbb{N}_0$ satisfies $d^*(N) \leq \delta$, then $H_N(U) \cap H_{\varphi^+}(U) = \{0\}$.

2. If $\Omega$ is spherically open and if $d^*(N_{\varphi^+}) \leq \delta$, then $T_{\varphi}^\pm$ is injective.

Proof.

1. We may assume that $0 \not\in N$. According to [18, pp. 562] (see also [10, p. 178]), there exists a set $X \subset [0, \infty) \setminus \mathbb{N}$ with $d(N \cup X) = \delta$. Since $X$ is countable, there exists a number $\sigma \in (0, 1)$ such that the set $X' := X + \sigma$ does not intersect the non-negative integers. Then $\{\mu_n : n \in \mathbb{N}\} := N \cup X'$ is still a superset of $N$ with $d(N \cup X') = \delta$ and $(N \cup X') \cap \mathbb{N}_0 = N$. We set

$$\Psi(\alpha) := \prod_{n=1}^{\infty} \left(1 - \frac{\alpha^2}{\mu_n^2}\right) \quad (\alpha \in \mathbb{C}).$$

Then $\Psi \in \text{Exp}(K_\delta)$ (see Example 5.4) with zeros exactly at the points $\pm \mu_n$. Hence, $\psi := M^{-1}\Psi \in H_{\varphi^+}(B_\delta^*)$ with $N_{\psi^+} = N$.

By assumption, there exists a spherically open set $W \subset U$ such that $B_\delta^* \ast W$ is connected. With Remark 2.2 and Theorem 5.1 we conclude

$$H_N(W) \cap H_{\varphi^+}(W) = \ker(T_{\psi}^+) \cap H_{\varphi^+}(W) \subset \ker(T_{\psi}^+) = \{0\}. $$

Because $U$ is a domain, we get $H_N(U) \cap H_{\varphi^+}(U) = \{0\}$.

2. With Remark 2.2 and 1. we obtain

$$\ker(T_{\varphi}^+) \subset H_{N_{\varphi^+}}(U) \cap H_{\varphi^+}(U) = \{0\}, $$

\[\square\]

5.6 Remark. 1. If we have $\delta = 0$ in Theorem 5.5, then we can choose $U = W$ and thus the assertions hold for all spherical domains $U$. The first assertion can be interpreted in the following way:
Whenever a power series about zero whose non-vanishing coefficients have density zero can be analytically continued up to infinity, then the power series must vanish. Interpreted this way, the assertion is a special case of the Fabry gap theorem (see e.g. [9, Section 11.7], [20, Section 6.4]).

2. If $U$ contains a keyhole domain $W$ of the form

$$W_{r,R}(\eta) = \{0 < |z| < r\} \cup \{|z| > R\} \cup \{z : |\arg(z)| < \pi\eta\}$$

for some $\eta > \delta$ (and some $0 < r < R < \infty$), then $B_\delta^* W_{r,R}(\eta) = W_{r,R}(\eta - \delta)$ is again a keyhole domain of the above form and thus in particular a spherical domain. The first assertion of Theorem 5.5 shows that whenever a power series about zero whose non-vanishing coefficients have maximal density $\delta$ can be analytically continued into a keyhole domain of the above form with $\eta > \delta$, then the power series must vanish. Similarly as in 1. this can now be seen as a special case of the Pólya gap theorem (see [18], [11, p. 3]).

3. According to symmetry (replace $\varphi$ by $\varphi(1/z)/z$) the conditions on $N^+\varphi$ can be replaced by the same conditions on $N^-\varphi$.

4. For examples highlighting the relation between Theorem 5.1 and Theorem 5.5 we refer to [14].

5.7 Corollary. Let $U \subset \mathbb{C}_*$ be a spherical domain. Each of the following conditions is sufficient for injectivity of $T^\pm \varphi$:

1. $d(N^+\varphi) = 0$

2. $d^*(N^-\varphi) = \delta < 1$ and $U$ contains a keyhole domain $W_{r,R}(\eta)$ for some $\eta > \delta$ (and some $0 < r < R < \infty$).

As a dual version of Theorem 5.5 we get

5.8 Theorem. Let $\Omega$ be spherically open and let $U \subset \mathbb{C}_*$ be open and so that $\Omega^* U^*$ has simply connected components. If $\delta := d^*(N^+\varphi) < 1$ then $T^\varphi$ has dense range if every open set $V \supset \Omega^* U^*$ contains a spherically open $W$ such that $B_\delta^* W$ is connected.

Proof. We show that the transposed operator is injective. Let $[(g,V)](\Omega^* U^*)_{\pm} \in H((\Omega^* U^*)_{\pm})$ with $[(g * \varphi,V * \Omega)](U^*)_{\pm} = [0](U^*)_{\pm}$. Then $g \in H_{N^+\varphi}(V_{\pm})$. Since $\Omega^* U^*$ is connected we can choose $V$ to be connected too. Then Theorem 5.5 (with $V$ instead of $U$) shows that $g = 0$ on $V$ and thus $T^\varphi$ is injective. \qed

5.9 Corollary. Let $\Omega$ be spherically open and let $U$ be open and so that $\Omega^* U$ has simply connected components. Each of the following conditions is sufficient for $T^\varphi$ to have dense range

1. $d(N^+\varphi) = 0$.

2. $d^*(N^\varphi) = \delta < 1$ and $\Omega * U$ omits a closed cone $\{z : |\arg(z)| \geq \pi(1 - \delta)\}$ of opening $2\pi\delta$.  

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6 Surjectivity of $T_\varphi$

We now turn towards the question under which conditions the operator $T_\varphi$ is even surjective. We start with the special case of $\varphi$ being the preimage of a polynomial under the Mellin transform.

6.1 Theorem. Let $\Phi$ be a non-vanishing polynomial and $\varphi = M^{-1}\Phi$. Then $T_\varphi$ is surjective for every open set $U \subset \mathbb{C}_*$ with simply connected components.

Proof. Without loss of generality we can assume $U$ to be simply connected (otherwise apply the subsequent argumentation to every component of $U$ separately). Moreover, we can assume that $\Phi$ is not constant.

If $\Phi$ is of degree $n \in \mathbb{N}$ and $c \in \mathbb{C} \setminus \{0\}$ is the leading coefficient, we set

\[
Z(\Phi) =: \{\lambda_j : j = 1, \ldots, n\}, \quad \Phi_j(\alpha) := \alpha - \lambda_j (\alpha \in \mathbb{C}, \ j \in \{1, \ldots, n\})
\]

and we obtain

\[
\Phi(\alpha) = c \prod_{j=1}^n \Phi_j(\alpha) (\alpha \in \mathbb{C}).
\]

Then for every $j \in \{1, \ldots, n\}$ we have $\varphi_j := M^{-1}\Phi_j = \kappa - \lambda_j 1_s$, where $\kappa$ denotes the Koebe function. The associative law for the convolution product shows that

\[
T_\varphi = c \cdot T_{\varphi_n} \circ \cdots \circ T_{\varphi_1}.
\]

Therefore, it is sufficient to show that for each $\lambda \in \mathbb{C}$ the operator $T_{\kappa-\lambda 1_s}$ is surjective.

Let $g \in H(U)$ be given. We have to find a function $f \in H(U)$ with $zf'(z) = \lambda f(z) + g(z)$ ($z \in U$). We fix a number $z_0 \in U$ and set

\[
f(z) := \exp(\lambda \log_U z) \cdot \int_{\gamma_z} \exp(-\lambda \log_U \zeta) \frac{g(\zeta)}{\zeta} d\zeta
\]

where $\gamma_z$ is a path in $U$ joining $z_0$ and $z$. Since $U$ is simply connected, this function is (well defined and) holomorphic in $U$ and solves the equation. $\Box$.

Before we come to a more general surjectivity criterion we need some auxiliary results. Let $K \subset \mathbb{C}$ be non-empty, compact and convex. Then the function

\[
H_K : \mathbb{C} \to \mathbb{C}, \quad H_K(z) := \sup_{u \in K} \text{Re}(zu),
\]

is called the support function of $K$. For $K, L \subset \mathbb{C}$ non-empty, compact and convex we have (see [2, pp. 64], [3])

1. $H_{K+L} = H_K + H_L$.

2. $K$ is a subset of $L$ if and only if $H_K \leq H_L$.

6.2 Lemma. Let $\Omega = (e^K)^*$, where $K \subset \mathbb{S}$ is compact and convex. Suppose furthermore $W \subset \mathbb{S}$ to be open and convex with $K + W \subset \mathbb{S}$ and let $U := e^{K+W}$. Then $\Omega \ast U = e^W$. 

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Proof. Since $U = e^{K+W} = e^W \cdot \Omega^*$, (1) implies that $\Omega \ast U$ is a superset of $e^W$. 
To obtain the reverse inclusion, we show $(e^W)^* \subset (\Omega \ast U)^*$. We have

$$(\Omega \ast U)^* = \Omega^* \cdot U^* = e^K \cdot (e^W+K)^* = e^K \cdot e^{-C \setminus O}$$

where $O := \bigcup_{k \in \mathbb{Z}} (W + K + 2k\pi i)$ (note that the sets $W + K + 2k\pi i$ ($k \in \mathbb{Z}$) are pairwise disjoint).

Now let $z \in (e^W)^*$. Then there is a point $v \in \mathbb{S} \setminus W$ such that for all $u \in K$ we have

$$z = e^{-v} = e^u \cdot e^{-(v+u)}.$$ 

If $u \in K$ can be chosen in such a way that $v + u \notin O$ we are done. Assume that this is not the case, i.e. $v + K \subset O$. Since $v + K$ is connected, it has to lie entirely in one component of $O$ and that component shall without loss of generality be the set $W + K$ itself. Since $W$ is convex, without loss of generality we can choose an exhaustion $(L_n)_{n \in \mathbb{N}}$ of $W$ (i.e. $(L_n)$ has the properties as in [19, Theorem 13.3]) consisting of convex sets. Then $(L_n + K)$ is an exhaustion of $W + K$ consisting of convex (and compact) sets. Since $v + K$ is compact, there is an integer $n_0$ such that $v + K \subset L_{n_0} + K$. Moreover, since in the latter inclusion all occurring sets are compact and convex, we can deduce $v \in L_{n_0} \subset W$ (which follows from the above properties of the support function). This contradicts the choice of $v$. 

\[ \square \]

6.3 Remark. In the situation of Lemma 6.2, the assumption $W + K \subset \mathbb{S}$ is crucial: Let $K = K_{1/2} = i[-\pi/2, \pi/2]$ and

$$W_1 = \{ w : \text{Im}(w) < \pi/2 \}, \quad W_2 = \{ w : |\text{Im}(w)| < \pi \} (= \mathbb{S}).$$

Then $W_1 + K = \mathbb{S}$ while $W_2 + K$ is a proper superset of $\mathbb{S}$. As a matter of fact, $\Omega \ast e^{W_1 + K} = e^{W_1}$, but $\Omega \ast e^{W_2 + K} = \mathbb{C}_+ \cdot e$ is a proper superset of $e^{W_2} = \mathbb{C}_-$. 

As already mentioned above, for a compact and convex set $L \subset \mathbb{C}$ we denote by $\text{Exp}(L)$ the space of all entire functions $\Phi$ of exponential type whose conjugate indicator diagram (denoted by $K(\Phi)$) is contained in $L$. According to [3, p. 74], the conjugate indicator diagram of $\Phi$ if the smallest closed convex set outside which the Borel transform of $\Phi$ has a holomorphic extension.

If $\Phi \neq 0$ is an entire function of exponential type, then $\Phi$ is said to be of \textit{completely regular growth} if there exists a set $E$ of relative zero (Lebesgue) measure such that $\lim_{\theta \rightarrow \pi} \frac{-1}{r} \log |\Phi(re^{it})|$ exists uniformly in $t \in [-\pi, \pi]$. With these notations we have (see [3, pp. 75]).

1. $K(\Phi + \Psi) \subset \text{conv}(K(\Phi) \cup K(\Psi))$ and $K(\Phi \Psi) \subset K(\Phi) + K(\Psi)$.

2. If $\Psi$ is of completely regular growth and $\Phi$ be an arbitrary entire function of exponential type then

$$K(\Phi \Psi) = K(\Phi) + K(\Psi). \quad (15)$$
We obtain the following sufficient criterion for the operator $T_\varphi$ to be surjective, where we assume that the functions $q_{k,\alpha} := q_{k,\alpha,U}$ are induced by the principal branch of the logarithm on $\mathbb{C}_-$.

6.4 Theorem. Let $\Phi$ be an entire function of completely regular growth with $K(\Phi) := K \subset \mathbb{S}$. Furthermore, suppose $W \subset \mathbb{S}$ to be open and convex with $W + K \subset \mathbb{S}$. If $U := e^W + K$ and $\varphi := M^{-1}\Phi$, then

$$\text{span}\{q_{k,\alpha} : \alpha \text{ m--fold zero of } \Phi, \ k \leq m - 1\}$$

is dense in $\ker(T_\varphi)$ and $T_\varphi : H(U) \to H(e^W)$ is surjective.

Proof.  
1. Due to Lemma 6.2 we have $\Omega * U = e^W$. Furthermore, according to Theorem 3.3,

$$Q := \text{span}\{q_{k,\alpha,U} : \alpha \text{ m--fold zero of } \Phi, \ k \leq m - 1\}$$

is a subset of $\ker(T_\varphi)$ (note that $U$ is simply connected).

In order to prove the denseness in $\ker(T_\varphi)$, using the Hahn-Banach theorem it suffices to show that

$$Q^\perp := \{u \in H'(U) : u(q) = 0 \text{ for all } q \in Q\} = \{0\}.$$ 

Let $u \in Q^\perp$ be given. According to Remark 4.2 there exists a unique germ $[(g,V)](U^*)$ such that

$$u(f) = (g * f)(1) \quad (f \in H(U)).$$

Since $U^* = \mathbb{C}_* \setminus e^{-(W + K)}$ and $W + K$ is convex, without loss of generality we can choose $V$ to be of the form $V = (e^{L+K})^*$ for some convex and compact set $L \subset W$. Therefore we can consider the Mellin transform $G := Mg \in \text{Exp}(L+K)$ of $g$.

If $\alpha \in \mathbb{C}$ is an $m-$fold zero of $\Phi$ and $k \leq m - 1$, we obtain with Remark 4.2 and Theorem 3.2

$$0 = u(q_{k,\alpha}) = (g * q_{k,\alpha})(1) = p_{\alpha,V^*U}(1) \sum_{l=0}^{k} \binom{k}{l} \log_{V^*U}^{k-l}(1)G^{(l)}(\alpha) = G^{(k)}(\alpha)$$

because $\log_{V^*U} 1 = 0$ (note that the proof of Theorem 3.2 reveals that the branch of the logarithm on $V^*U$ that fits to the principal branch of the logarithm on $\mathbb{C}_-$ fulfills $\log_{V^*U} 1 = 0$).

Hence, $\alpha$ is a zero of $G$ with multiplicity at least $m$ which implies that $G = \Psi \cdot \Phi$ for some entire function $\Psi$ and [1, Corollary 4.5.7] yields that $\Psi$ is of exponential type. Using (15) we obtain

$$H_K(\Psi) = H_K(G) - H_K(\Phi) \leq H_{L+K} - H_K = H_L$$

and thus $K(\Psi) \subset L$. This shows $\Psi \in \text{Exp}(L)$ and therefore $\psi := M^{-1}\Psi \in H((e^L)^*)$. Now, (6) implies that

$$M(\varphi * \psi) = \Phi \cdot \Psi = G$$

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and since $M$ is bijective it follows that $\varphi \ast \psi = g$.

For a given $f \in \ker(T_\varphi)$ we obtain with the associative law

$$u(f) = (g \ast f)(1) = ((\varphi \ast \psi) \ast f)(1) = (\psi \ast (\varphi \ast f))(1) = 0.$$  

This completes the proof of (16).

2. In order to prove the surjectivity of $T_\varphi$ we note that Theorem 3.5 yields that $\text{im}(T_\varphi)$ is dense in $H(\Omega \ast U)$. If we manage to show that $\text{im}(T_\varphi)$ is closed in $H(\Omega \ast U)$, the proof will be complete. In order to do that, the closed range theorem (see e.g. [15]) and 1. ensure that it is enough to show that

$$\text{im}(T_\varphi') = Q^\perp.$$ 

Since $Q \subset \ker(T_\varphi)$ it is clear that the left-hand side is a subset of the right-hand side. If, on the other hand, $u \in Q^\perp$, then we have shown above that the corresponding germ $[(g, V)]_{(U^\ast)_\pm}$ can be written as

$$[(g, V)]_{(U^\ast)_\pm} = [(\varphi \ast \psi, V)]_{(U^\ast)_\pm}$$

for some suitable $[(\psi, (e^{\ell^2})^\ast)]_{(\Omega^\ast U^\ast)_\pm} \in H((\Omega^\ast U^\ast)_\pm)$. Hence, the corresponding functional $v \in H(\Omega \ast U)^\ast$ fulfills $T_\varphi' v = u$ and we obtain $u \in \text{im}(T_\varphi')$. □

6.5 Remark. The assertion of Theorem 6.4 can be considered as a special case of a result concerning the surjectivity of operators which are defined via a convolution of an analytic functional with a holomorphic function formulated in [2, Prop. 1.5.12]. For more details we refer to [14]. Actually, the proof of Theorem 6.4 runs along the same lines as the proof of [2, Prop. 1.5.12].

6.6 Example. We consider again the situation in Example 5.4.1. There we stated that $T_\varphi$ has dense range. Actually the function $\Phi$ is of completely regular growth with $K(\Phi) = K_\delta$ (see e.g. [13], p. 205). Hence, Theorem 6.4 (with $W = \{ w : |\text{Im}(w)| < \pi (1 - \delta) \}$) yields that the operator $T_\varphi$ is even surjective.

Since each function of exponential type zero is of completely regular growth (see e.g. [2, p. 90], [13, p. 158]), we obtain in particular

6.7 Corollary. Let $\Phi \neq 0$ be of exponential type 0 and let $U \subset \mathbb{C}_-$ be a simply connected domain so that $\log U$ is convex. Then

$$\text{span}\{q_{k,\alpha} : \alpha \text{ m-fold zero of } \Phi, k \leq m - 1\}$$

is dense in $\ker(\Phi(\vartheta))$ and $\Phi(\vartheta)$ is surjective.

6.8 Remark. Similar results for corresponding classical differential operators of infinite order $\Phi(D) : H(G) \to H(G)$,

$$\Phi(D)h = \sum_{k=0}^{\infty} \Phi_k D^k h \quad (h \in H(G)),$$

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where $Dh := h'$ and $G \subset \mathbb{C}$ a domain, are well-known (see e.g. [2, Theorem 6.4.4]). Actually, the operators $\Phi(D)$ are closely related to the Euler differential operators $\Phi(\vartheta)$: If $U \subset \mathbb{C}^-$ is a domain and $G := \log(U)(\subset \mathbb{S})$, then

$$(\Phi(D)h) \circ \log = \Phi(\vartheta)(h \circ \log) \quad (h \in H(G))$$

which means that the following diagram commutes:

$$
\begin{array}{ccc}
H(G) & \xrightarrow{\Phi(D)} & H(G) \\
\circ \exp_{|G|} & \circ \log_{U} & \circ \exp_{|G|} \circ \log_{U} \\
H(U) & \xrightarrow{\Phi(\vartheta)} & H(U)
\end{array}
$$

This implies that results for the classical and the Euler differential operators turn out to be equivalent, if $G \subset \mathbb{S}$.

6.9 Remark. Theorem 6.1 shows that for a polynomial $\Phi$ the convexity of $\log U$ is not necessary for surjectivity of $\Phi(\vartheta)$. On the other hand, for transcendental $\Phi$ convexity of $G$ turns out to be necessary for surjectivity of $\Phi(D)$ (see [12]). According to Remark 6.8 the same is true for $\Phi(\vartheta)$.

6.10 Remark. In the above results we were concerned with the surjectivity of $T_{\varphi}$. For $U_+ = \{z : |z| < 1\}$ and $\Omega = \mathbb{C}^+ \setminus \{z = 1\}$ it is easily seen that the "one-sided" operator $T_{\varphi}^+$ turns out to be surjective if and only if $N_{\varphi^+} = \emptyset$ and $|\varphi^+|^{1/\nu} \to 1$ as $\nu \to \infty$. In [7], Frerick provides characterizations of the surjectivity of $\Phi(\vartheta)^+: H^+(U) \to H^+(U)$ for more general $U$:

1. $\Phi(\vartheta)^+$ is surjective, for all $U$ so that $U_+$ is simply connected, if and only if $Z(\Phi) \cap \mathbb{N}_0 = \emptyset$ and $\Phi$ is a polynomial or

$$\lim_{\alpha \to \infty, \alpha \in Z(\Phi)} \alpha/|\alpha| = -1.$$

2. $\Phi(\vartheta)^+$ is surjective, for all $U$ so that $U_+$ is starlike with respect to the origin, if and only if $Z(\Phi) \cap \mathbb{N}_0 = \emptyset$ and $\Phi$ is a polynomial or

$$\limsup_{\alpha \to \infty, \alpha \in Z(\Phi)} \Re(\alpha/|\alpha|) \leq 0.$$

References


