Series expansions for the computation of Bessel functions of variable order on bounded intervals

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Abstract. Based on a continuity property of the Hadamard product of power series we derive results concerning the rate of convergence of the partial sums of certain polynomial series expansions for Bessel functions. Since these partial sums are easily computable by recursion and since cancellation problems are considerably reduced compared to the corresponding Taylor sections, the expansions may be attractive for numerical purposes. A similar method yields series expansions for confluent hypergeometric functions.

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1. Introduction

Consider the Bessel functions of first kind $J_\lambda$ given by

$$J_\lambda(w) := \frac{(w/2)^\lambda}{\Gamma(\lambda + 1)} f_\lambda(-w^2),$$

where $\lambda \in \mathbb{C} \setminus (-\mathbb{N})$ and

$$f_\lambda(z) := \sum_{\nu=0}^{\infty} \frac{z^\nu}{(\lambda + 1)_\nu \nu! 4^\nu}, \quad (z \in \mathbb{C})$$

with Pochhammer’s notation $(\zeta)_\nu := \prod_{k=1}^{\nu} (\zeta + k - 1)$ and $(\zeta)_0 := 1$.

In standard routines for the evaluation of Bessel functions of arbitrary order $\lambda$, partial sums of (2) are taken as approximations of $f_\lambda$ in a certain neighbourhood of the origin (cf. [?]). In the (usual) case of fixed precision computation, the size of the neighbourhood is (in particular for $\lambda$ of small modulus) seriously restricted due to cancellation errors occurring in the evaluation of the Taylor sections of (2).

In the real variable case, cancellation problems may be reduced by using (shifted) Chebyshev sections on certain intervals instead of Taylor sections, for which also the rate of convergence is better. However, since the effort for the computation of Chebyshev coefficients is much higher than the effort for the computation of Taylor coefficients (which may be easily computed by recursion), Chebyshev sections don’t provide an alternative in routines for parameter families as above, where the coefficients depend on the parameter and thus cannot be stored.

We want to invite series expansions for the functions $f_\lambda$ which have the advantage of being numerically more stable than Taylor sections and having a better rate of convergence, and which are computable easily by recursion.
If we denote for two (formal) power series \( \varphi(z) = \sum_{\nu=0}^{\infty} \varphi_{\nu} z^{\nu} \) and \( \psi(z) = \sum_{\nu=0}^{\infty} \psi_{\nu} z^{\nu} \) the Hadamard product of \( \varphi \) and \( \psi \) by

\[
(\varphi \ast \psi)(z) = \sum_{\nu=0}^{\infty} \varphi_{\nu} \psi_{\nu} z^{\nu},
\]

then the functions \( f_\lambda \) may be written in the form

\[
f_\lambda(z) = (\varphi_\lambda \ast g)(z) = \sum_{\nu=0}^{\infty} \frac{(\lambda_0 + 1)_{\nu}}{(\lambda + 1)_{\nu}} z^{\nu} \ast f_{\lambda_0}(z),
\]

where \( g = f_{\lambda_0} \) for some fixed \( \lambda_0 \), and

\[
\varphi_\lambda(z) = F(\lambda_0 + 1, 1; \lambda + 1; z)
\]

is a hypergeometric function. As usual, we denote by \( F(a, b; \cdot) \) the hypergeometric functions

\[
F(a, b; c; z) := \sum_{\nu=0}^{\infty} \frac{(a)_{\nu}(b)_{\nu}}{(c)_{\nu}} \nu! z^{\nu}.
\]

(and the analytic continuations). If

\[
g = \sum_{k} a_k T_k
\]

is some series expansion for \( g \) (for example, the shifted Chebyshev expansion on some compact interval), then we can expect the \( f_\lambda \) to be expandable as

\[
f_\lambda = \sum_{k} a_k (\varphi_\lambda \ast T_k).
\]

The crucial point is that the coefficients \( a_k \) no longer depend on the parameter \( \lambda \). Thus, if \( \varphi_\lambda \ast T_k \) is computable with only some small effort, for example by some recursion formula, then the expansion (5) may hopefully be used for the efficient evaluation of \( f_\lambda \).

2. Formal Relations and Theory

Since we are mainly interested in approximations of \( J_\lambda(x) \) for real arguments \( x \), we restrict our investigations to expansions of \( f_\lambda \) on intervals \( I_\alpha := [-\alpha^2, 0] \) for some \( \alpha > 0 \), which leads to approximations of \( J_\lambda \) on \( [-\alpha, \alpha] \) (or \( (0, \alpha] \)).

Let \( T_{k,\alpha} \) denote the shifted Chebyshev polynomial with respect to \( I_\alpha \), that is,

\[
T_{k,\alpha}(x) := t_k(1 + 2x/\alpha^2)
\]

where \( t_k \) is the \( k \)-th Chebyshev polynomial. Since

\[
T_{k,\alpha}(z) = F(-k; k; 1/2; -z/\alpha^2)
\]

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(see for example [?], p. 176), we get for \( \varphi \lambda \) as above

\[
(\varphi \lambda T_{k,\alpha})(z) = 3 F_2(-k, k, \lambda_0 + 1; \lambda + 1, 1/2; -z/\alpha^2) = \sum_{\nu=0}^{k} \frac{(-k)_\nu (k)\nu (\lambda_0 + 1)\nu \cdot z^\nu}{(\lambda + 1)\nu (1/2)\nu \nu!}.
\]  

(6)

The polynomials \( \varphi \lambda T_{k,\alpha} \) may be computed by a four-term recurrence formula which is given in [?], p. 147. Moreover, if we choose in particular \( \lambda_0 = -1/2 \), then the generalized hypergeometric function in (6) reduces to a classical hypergeometric function, i.e.

\[
(\varphi \lambda T_{k,\alpha})(z) = 2 F_1(-k, k; \lambda + 1; -z/\alpha^2) = F(-k, k; \lambda + 1; -z/\alpha^2).
\]

For \( \lambda \in (-1, 0) \) the polynomials

\[
T_k(\lambda, w) := F(-k, k; \lambda + 1, w)
\]

are connected to Jacobi polynomials (cf. [?], p. 167). Therefore, we may obtain from the three-term recurrence relation for Jacobi polynomials a three-term recurrence for \( T_k(\lambda, w) \) given by

\[
(2k - 1)(\lambda + k + 1)T_{k+1}(\lambda, w) = [(2k + 1)(2k - 1)(1 - 2w) - (1 + 2\lambda)]T_k(\lambda, w) + (2k + 1)(\lambda - 1 - k)T_{k-1}(\lambda, w)
\]

(7)

with \( T_0(\lambda, w) = 1 \) and \( T_1(\lambda, w) = 1 - w/(\lambda + 1) \). Now, since

\[
g(z) = f_{-1/2}(z) = \sum_{\nu=0}^{\infty} \frac{z^\nu}{(1/2)\nu!4^\nu} = \sum_{\nu=0}^{\infty} \frac{z^\nu}{(2\nu)!} \left( \equiv \cos \sqrt{-z} \right),
\]

we get from the Chebyshev expansion of \( \cos(\alpha x) \) (see for example [?], p. 53) the shifted Chebyshev expansion

\[
g(x) = J_0(\alpha) + 2 \cdot \sum_{k=1}^{\infty} J_{2k}(\alpha) T_{k,\alpha}(x) \quad (x \in I_\alpha).
\]

(8)

Therefore, we finally obtain the (for the moment formal) series representation

\[
f_\lambda(z) = (\varphi \lambda g)(z) = J_0(\alpha) + 2 \cdot \sum_{k=1}^{\infty} J_{2k}(\alpha) T_k(\lambda, -z/\alpha^2)
\]

(9)

for the entire functions \( f_\lambda \) in which the coefficients \( J_{2k}(\alpha) \) are independent of the parameter \( \lambda \). Hence, if these coefficients are stored, the evaluation of the partial sums of (9) may be easily performed by recursive computation of \( T_k(\lambda, -z/\alpha^2) \) according to (7).
Let $S_{n,\alpha}$ denote the $n$-th partial sum of (8) and let
\[
||\varphi||_K := \sup_{z \in K} |\varphi(z)|
\]
for a compact plane set $K$ and $\varphi$ being continuous on $K$. The following result shows that the errors of approximation for the partial sums $\varphi_\lambda * S_{n,\alpha}$ of (9) are independent of $\lambda$ for $\lambda \geq -1/2$ and $n$ large enough.

**Theorem 1.** If $\alpha > 0$ is given, then for every $\lambda \geq -1/2$ and every $n \geq \alpha/2 - 1$ we have
\[
||f_\lambda - \varphi_\lambda * S_{n,\alpha}||_{I_\alpha} = ||g - S_{n,\alpha}||_{I_\alpha} = 1 - S_{n,\alpha}(0) = 2 \sum_{k>n} J_{2k}(\alpha) .
\] (10)

For the proof we need

**Theorem 2.** Let $\psi$ be an arbitrary entire function and suppose that $\varphi = F(a,1;c;\cdot)$ is a hypergeometric function with $0 < a \leq c$. Then we have
\[
||\varphi * \psi||_K \leq ||\psi||_K
\]
for all compact plane sets which are starlike with respect to the origin (that is, $zt \in K$ for all $z \in K$ and all $0 \leq t \leq 1$).

A proof of Theorem 2 is found in [?].

**Proof of Theorem 1.**
1. We start with the second and third equalities. Let $\alpha > 0$ and $n \geq \alpha/2 - 1$ be given. From $|T_{k,\alpha}(x)| = |t_k(1 + 2x/\alpha^2)| \leq 1$ for all $x \in I_\alpha$ and $T_{k,\alpha}(0) = 1$ we immediately obtain (since $J_{2k}(\alpha) > 0$ for $k \geq \alpha/2$)
\[
||g - S_{n,\alpha}||_{I_\alpha} = 1 - S_{n,\alpha}(0) = 2 \sum_{k>n} J_{2k}(\alpha) .
\] (11)

2. The proof of $\leq$ in the first equality follows from Theorem 2 with $\varphi = \varphi_\lambda$ and $\psi = g - S_{n,\alpha}$. Now equality holds by 1. and the fact that $f_\lambda(0) - \varphi_\lambda * S_{n,\alpha}(0) = 1 - S_{n,\alpha}(0)$.

**Remarks.**
1. Since $f_n(w) \to 1$ for $n \to \infty$ and all $w \in \mathfrak{F}$, we may obtain from (10)
\[
||f_\lambda - \varphi_\lambda * S_{n,\alpha}||_{I_\alpha} \sim 2J_{2(n+1)}(\alpha) \sim \frac{1}{(2(n+1))!} \left(\frac{\alpha}{2}\right)^{2(n+1)} (n \to \infty) .
\] (12)
For the Taylor sections $S_n(f_\lambda)$ of $f_\lambda$ we have

$$||f_\lambda - S_n(f_\lambda)||_{I_\alpha} \sim \frac{1}{(\lambda + 1)n!} \left( \frac{\alpha}{2} \right)^{2(n+1)} \quad (n \to \infty). \quad (13)$$

Therefore, since $(2(n+1))! = (n+1)!(1/2)_{n+1/2}^n$, we get an asymptotic acceleration factor of $0.5 \cdot 4^{-n}(\lambda + 1)_{n+1}^{-1}/(1/2)_{n+1}^{-1}$.  

2. For arbitrary $\lambda \in C \setminus (-\mathcal{N})$ an inequality of the form (10) is no longer valid. However, since

$$||S_{n+1,\alpha} - S_{n,\alpha}||_{I_\alpha} = 2J_{2(n+1)}(\alpha),$$

similar considerations as in [?] show that, for every $\lambda$ and every $\delta > 1$, there exists a constant $M_\lambda(\delta) > 0$ such that

$$||f_\lambda - \varphi_\delta * S_{n,\alpha}||_{I_\alpha} \leq 2M_\lambda(\delta) \sum_{m>n} \delta^m J_{2m}(\alpha) \sim 2M_\lambda(\delta) \delta^{n+1} J_{2(n+1)}(\alpha) \sim \left( \frac{\alpha}{2} \right)^{2(n+1)} \quad (n \to \infty)$$

for $n \to \infty$. In particular, by Stirling’s formula and a result found in [?] or [?], this implies that the $n$-th root rate of convergence is, for all $\lambda$, asymptotically optimal in the sense that

$$\limsup_{n \to \infty} n^{2}||f_\lambda - \varphi_\delta * S_{n,\alpha}||_{I_\alpha}^{1/n} = \left( e\alpha/4 \right)^2 = \limsup_{n \to \infty} n^2 E_n(f_\lambda, I_\alpha)^{1/n},$$

where $E_n(f, K) := \inf_{P \in \Pi_n} ||f - P||_K$ denotes the error of the best approximating polynomial of degree $\leq n$ to $f$ on $K$. According to (13), this is not the case for the Taylor sections $S_n(f_\lambda)$.

As we have noted in the introduction, the main restriction in the use of Taylor sections $S_n(f_\lambda)$ results from cancellation errors occurring in the evaluation of $S_n(f_\lambda)$ for $\lambda$ of small modulus. For example, if $\lambda = 0$ and $k \in \mathcal{N}$, we have

$$J_0(2k) = f_0(-2k)^2 = \sum_{\nu=0}^{\infty} (-1)^\nu (k^\nu/\nu)!^2$$

and, by Stirling’s formula,

$$\max_{\nu \in \mathcal{N}} \left( \frac{k^\nu}{\nu!} \right)^2 \sim \left( \frac{k^k}{k!} \right)^2 \sim e^{2k}/2\pi k.$$

Since $|f_0(-x^2)| \leq 1$ for all $x$, we have to accept for example for $k = 10$ a loss of 7 decimal digits (note that $e^{20}(20\pi)^{-1} \approx 8 \cdot 10^6$). On the other hand, for $\lambda \geq -1/2$, in the expansion (9) the summands are not larger than 1 in modulus, since $|J_\nu(\alpha)| \leq 1$ for all $n \in \mathcal{N}$ and all $\alpha > 0$, and since $|T_k(\lambda, y)| \leq 1$ for all
\( y \in [0,1] \) and all \( \lambda \geq -1/2 \) (the last inequality follows from Theorem 2 since \( T_k(\lambda, \cdot) = F(1/2, 1; \lambda + 1; \cdot) \ast T_k(1/2, \cdot) \) and \(|T_k(1/2, y)| = |t_k(1 - 2y)| \leq 1 \) for \( y \in [0,1] \)). Thus, in cases where \( f_\lambda(x) \) is not too small in modulus, we have essentially the full accuracy.

The following figures show approximations of the number of significant decimal digits for \( f_\lambda(x) \), where \( x = 1(1)20 \) and \( \lambda = 0(1)39 \) (Figure 1) and \( \lambda = -0.9(0.1)1.0 \) (Figure 2), given by

\[
d_1(\lambda, x) := \min\{15, -\log_{10}(|f_\lambda(x) - S_{40}(f_\lambda(x))/|f_\lambda(x)|)|\}
\]

and

\[
d_2(\lambda, x) := \min\{15, -\log_{10}(|f_\lambda(x) - \varphi_\lambda \ast S_{25,20}(x))/|f_\lambda(x)|)|\}
\]

The computations were carried out in double precision FORTRAN which gives a maximal accuracy of about 16 decimal digits. Since the usual accuracy requirement for special functions routines is 15 decimal digits in double precision versions, we have cut off the errors at a level of 15 digits, so that all values on the 15-level represent approximations within the usual tolerance. The degrees \( n \) (which are 40 for Taylor sections and 25 for the sections of (9)) are chosen in such a way that the errors result exclusively from cancellation which implies that higher degrees do not reduce these errors.

Figure 1.
Remark. A similar argumentation as above applies to the family of confluent hypergeometric functions $K(a, c; \cdot)$ given by

$$K(a, c; z) := \sum_{\nu=0}^{\infty} \frac{(a)_\nu}{(c)_\nu \nu!} z^\nu.$$ 

Here we may write

$$K(a, c; \cdot) = F(a, 1; c; \cdot) * K(1, 1; \cdot) = F(a, 1; c; \cdot) * \exp(\cdot).$$

Thus, if $I_\beta$ denotes the interval between 0 and $\beta$ for $\beta$, and if

$$\exp(z) = \sum_{k=0}^{\infty} a_{k,\beta} T_{k,\beta}(z)$$

is the shifted Chebyshev expansion of $\exp$ on $I_\beta$, then we obtain from

$$T_{k,\beta}(z) = F(-k, k; 1/2; z/\beta)$$

the expansion

$$K(a, c; z) = \sum_{k=0}^{\infty} a_{k,\beta} \cdot \_3F_2(-k, k, a; c, 1/2; z/\beta)$$

As we have noted above, the polynomials $\_3F_2(-k, k, a; c, 1/2; z)$ may be computed recursively by a four-term recurrence relation found for example in [?], p. 147. Therefore, application of the convergence theory of [?] shows that the above series expansion provides a method for the numerical evaluation of confluent hypergeometric functions on the intervals $I_\beta$. 

Figure 2.
References

[1] D.E. Amos, S.L. Daniel and M.K. Weston, CDC 6600 Subroutines IBESS and JBESS for Bessel functions $I_\nu(x)$ and $J_\nu(x), x \geq 0, \nu \geq 0$, ACM Trans. Math. Software 3 (1977) 76-92.


