From Polynomial Approximation to Universal Taylor Series and Back Again

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pa2uTs For a compact set $E \subset \mathbb{C}$ let $(A(E), \|\cdot\|_E)$ denote the Banach space of all $f : E \to \mathbb{C}$ continuous on E and holomorphic in E^0 with the uniform norm. Moreover, let $P(E)$ be the closure of the polynomials in $A(E)$ and let $H(E)$ be the set of functions on E extending holomorphically to some neighborhood of E. For E^c connected, Runge's theorem says that $H(E) \subset P(E)$ and Mergelian's theorem shows that, more precisely, $A(E) = P(E)$.

For a domain $\Omega \subset \mathbb{C}$, which is always supposed to contain the unit disk $\mathbb{D}$ but not its closure, and for a function $f \in H(\Omega)$ we consider the Taylor sections

$$(S_n f)(z) = \sum_{\nu=0}^{n} \frac{f^{(\nu)}(0)}{\nu!} z^\nu \qquad (n \in \mathbb{N},\, z \in \mathbb{C})$$

and ask for results on

$$\omega(f, E) := \{g \in A(E) : \exists\, (n_m) : S_{n_m} f \to g \text{ in } A(E)\}.$$

For $K \subset \mathbb{D}^c$ compact with K^c connected it may happen that $\omega(f, K)$ is maximal. We set

$$U_K(\Omega) := \{f \in H(\Omega) : \omega(f, K) = A(K)\}.$$

Applying the universality criterion (see [GE]), we give a short proof of (cf. [Ne])

Proposition 1. *Let Ω be simply connected. Then $U_K(\Omega)$ is residual in $H(\Omega)$ for all compact sets $K \subset \Omega^c$ with K^c connected.*

Proof. We consider $S_n : H(\Omega) \to A(K)$ with the compact-open topology on $H(\Omega)$. Then the S_n are continuous. From Mergelian's theorem it follows that, for a suitable sequence of polynomials (h_j), the sequence $V_j := \{\psi \in A(K) : \|\psi - h_j\| < 1/j\}$ forms a countable base of the topology of $A(K)$. According to the universality criterion we have to guarantee that for all $j \in \mathbb{N}$, all compact sets $L \subset \Omega$ with L^c connected and all $g \in H(\Omega)$ there are arbitrary large $n \in \mathbb{N}$ with $S_n(\{\varphi \in H(\Omega) : \|\varphi - g\|_L < 1/j\}) \cap V_j \neq \emptyset$.

Let $E := L \cup K$. Since $f : E \to \mathbb{C}$ with $f_{|L} := g$ and $f_{|K} := h_j$ is in $H(E)$, and since E has connected complement, Runge's theorem offers a polynomial p with $\|p - g\|_L < 1/j$ and $p \in V_j$. Noting that $S_n(p) = p$ for all $n \geq \deg(p)$ we are done. $\square$

We remark that the proof equally works for arbitrary sequences $T_n : H(\Omega) \to A(K)$ of continuous projections to the set of polynomials of degree $\leq n$, as e. g. Faber sections or sequences of suitable interpolating polynomials.

By using variants of Runge's theorem it is possible to impose further conditions on universal Taylor series. We consider lacunary series: For $\Lambda \subset \mathbb{N}_0$ let

$$H_\Lambda(\Omega) := \{f \in H(\Omega) : f^{(\nu)}(0) = 0\, (\nu \notin \Lambda)\},\ U_{K,\Lambda}(\Omega) := H_\Lambda(\Omega) \cap U_K(\Omega)$$

and let $P_\Lambda(E)$ be the closed linear span of the monomials $z \mapsto z^\nu$ ($\nu \in \Lambda$) in $A(E)$. If $0 \in E^0$, then $f \in P_\Lambda(E)$ implies $f^{(\nu)}(0) = 0$ for all $\nu \notin \Lambda$. Conversely, we have the following Runge type result (see [LMM]):

Suppose that E is compact with E^c connected, $0 \in E^0$ and such that the component of E containing 0 is starlike with respect to 0. If Λ has upper density $\bar{d}(\Lambda) = 1$, then every $f \in H(E)$ with $f^{(\nu)}(0) = 0$ for all $\nu \notin \Lambda$ is in $P_\Lambda(E)$.

Since no extra conditions are imposed on components of E not containing 0, the same proof as for Proposition 1 gives (cf. [Sch])

Proposition 2. *Let Ω be starlike with respect to 0 and suppose that $\bar{d}(\Lambda) = 1$. Then $U_{K,\Lambda}(\Omega)$ is residual in $H_\Lambda(\Omega)$ for all compact sets $K \subset \Omega^c$ with K^c connected.*

Remarks 3. By topological arguments and a further application of Mergelian's theorem (only on the "K-side") it can be shown (see [Ne]) that for Ω simply connected there is a sequence (K_j) in Ω^c with K_j^c connected and

$$U(\Omega) := \bigcap \{U_K(\Omega) : K \subset \Omega^c,\ K^c \text{ connected}\} = \bigcap_{j \in \mathbb{N}} U_{K_j}(\Omega)$$

Therefore, $U(\Omega)$ is still residual in $H(\Omega)$. The same arguments lead to the residuality of $U_\Lambda(\Omega) := \bigcap \{U_{K,\Lambda}(\Omega) : K \subset \Omega^c,\ K^c \text{ connected}\}$ in $H_\Lambda(\Omega)$ for Ω starlike and $\bar{d}(\Lambda) = 1$.

From a result in [MM], it follows that the condition $\bar{d}(\Lambda) = 1$ turns out to be sharp. More precisely, given $d < 1$, there is a compact sector $S_d \subset \mathbb{D}^c$ such that for all K with $K^0 \supset S_d$ and all $f \in H_\Lambda(\mathbb{D})$ with $\bar{d}(\Lambda) \leq d$ the condition $0 \in \omega(f, K)$ implies $f \equiv 0$. Therefore, in particular, for Λ with $\bar{d}(\Lambda) < 1$ always $U_\Lambda(\mathbb{D}) = \emptyset$.

uTs2pa That polynomial approximation has impact on the existence of universal Taylor series is well known. On the other hand, universality properties of (S_n) lead to certain overconvergence and thus to extra approximation of f in $\Omega \setminus \mathbb{D}$. In [MY], the following result on reduced growth of sequences of polynomials is found:

Lemma 4. *Let $B \subset \mathbb{C}$ be closed and non-thin at ∞. If (p_m) is a sequence of polynomials with*

$$\limsup_{m \to \infty} |p_m(z)|^{1/d_m} \leq 1 \qquad (z \in B),$$

where deg$(p_m) \leq d_m$, then for all compact $E \subset \mathbb{C}$

$$\limsup_{m \to \infty} \|P_m\|_E^{1/d_m} \leq 1.$$

If $f \in H(\mathbb{D})$ is so that for some B as in the Lemma

$$\limsup_{m \to \infty} |S_{n_m}(z)|^{1/n_m} \leq 1 \qquad (z \in B),$$

an application of the two-constants-theorem (similarly as in the proof of the classical Ostrowski's overconvergence theorem, see e. g. [Hi], Theorem 16.7.2) shows that f has a maximal domain of existence Ω_f, that Ω_f is simply connected, and that $S_{n_m} f \to f$ in $H(\Omega_f)$ (thus $f \in \omega(f, L)$ for all $L \subset \Omega_f$ compact). In particular, this is satisfied if Ω is simply connected and $f \in U(\Omega)$ (the complement of a simply connected domain is non-thin at ∞). In this case, also $\Omega = \Omega_f$, that is, all $f \in U(\Omega)$ have Ω as natural boundary (cf. [MVY]).

From results in [Ge] it follows that some overconvergence of (S_n) already occurs under weaker conditions: If $f \in H(\Omega)$ for $\Omega \neq \mathbb{D}$ and if $\omega(f, E) \neq \emptyset$ for some compact set $E \subset \mathbb{C}$ with $\text{cap}(E) > 1$, then there is a domain $\Omega_E \not\supseteq \mathbb{D}$ with $S_{n_m} f \to f$ in $H(\Omega_E)$ for some (n_m).

On the other hand, Taylor sections of functions in $H(\mathbb{C} \setminus \{1\})$ cannot exhibit overconvergence (or, equivalently, cannot have Hadamard-Ostrowski gaps). This follows from the classical Wigert's theorem in connection with [Po], Theorem V. However, in [Me], it is shown that $U_K(\mathbb{C} \setminus \{1\})$ is residual in $H(\mathbb{C} \setminus \{1\})$, for all $K \subset \mathbb{D}^c$ finite.

In view of the above results a reasonable guess is that K finite might be replaced by $\text{cap}(K) = 0$.

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