Noname manuscript No.
(will be inserted by the editor)

Spurious limit functions of Taylor series

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Received: March 19, 2019/ Accepted:

Abstract It is known that, generically in the space $H(\mathbb{D})$ of functions holomorphic in the unit disc $\mathbb{D}$, the sequences $(S_n f)$ of partial sums of Taylor series behave extremely erratically on the unit circle $\mathbb{T}$. According to a result of Gardiner and Manolaki, the situation changes in a significant way if $f \in H(\mathbb{D})$ has nontangential limits on subsets of $\mathbb{T}$ of positive arc length measure. In this case each convergent subsequence tends to the nontangential limit function almost everywhere. We consider the question to which extent in spaces of holomorphic functions where nontangential limits are guaranteed, "spurious" limit functions, that is, limit functions different than the nontangential limit may appear on small subsets of $\mathbb{T}$.

Keywords nontangential limits · universal functions · boundary behaviour

Mathematics Subject Classification (2010) 30K05 (primary) · 30B30 · 30H10

1 Introduction

Let $\mathbb{C}_\infty$ be the extended complex plane. For an open set $\Omega \subset \mathbb{C}_\infty$ we denote by $H(\Omega)$ the Fréchet space of functions holomorphic in Ω (and vanishing at ∞ if $\infty \in \Omega$) endowed with the topology of locally uniform convergence. If $0 \in \Omega$ and $f \in H(\Omega)$ we write

$$(S_n f)(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$$

with $a_\nu = a_\nu(f) = f^{(\nu)}(0)/\nu!$ for the n-th partial sum of the Taylor expansion $\sum_{\nu=0}^\infty a_\nu z^\nu$ of f about 0. A classical question in complex analysis is how the

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partial sums $S_n f$ behave outside the disc of convergence and in particular on the boundary of the disc.

Let $\mathbb{D}$ and $\mathbb{T}$ denote the unit disc and the unit circle, respectively. It is known that, generically in $H(\mathbb{D})$, the behaviour of the sequence $(S_n f)$ on $\mathbb{T}$ is extremely erratic in the sense that all continuous functions on $\mathbb{T}$ are realised as pointwise limit functions of some subsequence of $(S_n f)$. For precise definitions and a large number of corresponding results the expository articles [15] and [20] are highly recommended. For results on universal series in a more general framework see also [1].

According to a result of Gardiner and Manolaki, the situation changes in a significant way if $f \in H(\mathbb{D})$ has nontangential limits on subsets of $\mathbb{T}$ of positive arc length measure. We write $f_{\triangleright}(\zeta)$ for the nontangential limit of f at the point $\zeta \in \mathbb{T}$ in case of existence. It turns out that there is a clear preference for the limit function $f_{\triangleright}$:

Theorem 1 (Gardiner and Manolaki, 2016)
Let $f \in H(\mathbb{D})$ and suppose that a subsequence of $(S_n f)$ converges to some function g pointwise on $L \subset \mathbb{T}$. If $f_{\triangleright}$ exists on L then g coincides with $f_{\triangleright}$ almost everywhere on L.

The proof given in [16] is based on advanced tools and methods from potential theory. It follows e.g. from a result of Costakis (see [9]) that the nontangential limits cannot be replaced by radial limits.

Theorem 1 is in a sense the benchmark for the investigations in this paper. We address the question to which extent limit functions different from $f_{\triangleright}$ may appear on small subsets of $\mathbb{T}$ where the existence of the nontangential limit function $f_{\triangleright}$ is guaranteed. In the sequel we say that a limit function $g : L \to \mathbb{C}$ of $(S_n f)$ on $L \subset \mathbb{T}$ is *spurious* if $g(\zeta) \neq f_{\triangleright}(\zeta)$ for all $\zeta \in L$.

We fix some more notations. For K compact in $\mathbb{C}$ and $M \subset \mathbb{C}$, let $C(K, M)$ be the set of continuous functions $h : K \to M$. As usual $C(K) := C(K, \mathbb{C})$ is endowed with the uniform norm with respect to K, denoted by $\|\cdot\|_K$. Then $A(K)$, the subspace of all $f \in C(K)$ holomorphic in the interior of K, is closed in $C(K)$. Finally, we say that a property is satisfied for *generically all elements* in a Baire space if it is satisfied on a dense G_δ subset of the space.

2 Spurious limit functions and analytic continuation

We start with the geometric series

$$\gamma(z) := 1/(1-z) \qquad (z \in \mathbb{C} \setminus \{1\}),$$

where

$$(S_{n-1}\gamma)(z) = \sum_{\nu=0}^{n-1} z^\nu = \gamma(z)(1-z^n).$$

Obviously, here a subsequence $(S_{n_j-1})_j$ converges uniformly on the closed set $E \subset \mathbb{T} \setminus \{1\}$ if and only $(\zeta^{n_j})_j$ converges uniformly on E to some function. A closed set $E \subset \mathbb{T}$ is called a *Dirichlet set* if for some infinite set $\Lambda \subset \mathbb{N}$ the subsequence $(\zeta^n)_{n\in\Lambda}$ of (ζ^n) is uniformly convergent on E to a limit function α (which then belongs to $C(E,\mathbb{T})$).[1] In this case, we also say that E is a (Λ,α)-Dirichlet set. If each function $\alpha \in C(E,\mathbb{T})$ is the uniform limit of some subsequence of (ζ^n) on E, then E is said to be a *Kronecker set*. Finally, a set $E \subset \mathbb{T}$ is called *pseudo Dirichlet*, if E is the union of an increasing sequence (E_m) of (Dirichlet) sets with the property that a subsequence of (ζ^n) converges uniformly on E_m for all m. It turns out that each countable union of increasing Dirichlet sets is a pseudo Dirichlet set (see [6, p. 357]).

Each finite set in $\mathbb{T}$ is Dirichlet and thus all countable sets in $\mathbb{T}$ are pseudo Dirichlet. Moreover, it can be shown that Dirichlet sets (and even Kronecker sets) of full Hausdorff dimension 1 exist; see [20, Section 6]. On the other hand, pseudo Dirchlet sets are small in the sense that they are sets of vanishing arc length measure (which follows in particular from Theorem 1 applied to γ).

With these notions, we see that $(S_n\gamma)$ has a uniform limit function on the closed set $E \subset \mathbb{T}\setminus\{1\}$ if and only if E is Dirichlet. In this case, each limit function is spurious and of the form $\gamma(1-\alpha)$ for some $\alpha \in C(E,\mathbb{T})$. If E is a Kronecker set then all such functions are limit functions.

In order to get a more complete picture, we consider general domains $\Omega \subset \mathbb{C}_\infty$ with $\mathbb{D} \subset \Omega$ and the Taylor shift $T : H(\Omega) \to H(\Omega)$, defined for $f \in H(\Omega)$ and $a_\nu = f^{(\nu)}(0)/\nu!$ by

$$(Tf)(z) := \begin{cases} (f(z)-a_0)/z, & z \neq 0 \\ a_1, & z = 0 \end{cases}.$$

It is easily seen that T is a continuous operator on $H(\Omega)$. Moreover, the n-th iterate T^n is given by

$$(T^n f)(z) := \begin{cases} (f - S_{n-1}f)(z)/z^n, & z \neq 0 \\ a_n, & z = 0 \end{cases}.$$

This indicates the intimate relation between the dynamical behaviour of T and the limit behaviour of $(S_n f)$ for $f \in H(\Omega)$. We consider the case of functions analytically continuable beyond $\mathbb{D}$ (which means $\Omega \neq \mathbb{D}$) and we write

$$R_n f := f - S_{n-1} f$$

for $f \in H(\Omega)$. Then we have

$$(R_n f)(\zeta) = \zeta^n (T^n f)(\zeta) \quad (\zeta \in \Omega \cap \mathbb{T}) \tag{1}$$

and in particular

$$|R_n f| = |T^n f| \quad \text{on } \Omega \cap \mathbb{T}. \tag{2}$$

[1] The usual definition requires convergence to the limit function 1. It is easily seen that the two notions agree: If a subsequence $(\zeta^{n_j})_j$ tends to some limit function α (uniformly on E) then $(\zeta^{n_{2k}-n_k})_k$ tends to $\alpha\overline{\alpha} = 1$.

It is easily seen that a non-zero function $f \in H(\Omega)$ is periodic for T with period k, i.e. $T^k f = f$, if and only if $\mathbb{C}_\infty \setminus \Omega$ contains k-th roots of unity and f is of the form $f(z) = p(z)/(1 - z^k)$ for some polynomial p of degree less than k. According to (1), in this case the situation concerning spurious limit functions is similar to the special case of the geometric series γ. Moreover, (1) indicates why in this case, for fixed $\zeta \in \mathbb{T} \cap \Omega$, the set of limit points of the sequence $((S_n f)(\zeta))$ has circular structure with centre $f(\zeta)$ (see e.g. [21], [22]).

We consider more general $f \in H(\Omega)$. The following result is essentially a reformulation of classical theorems of Fatou and M. Riesz.

Theorem 2 *Let Ω be a domain with $\mathbb{D} \subset \Omega$ and let $f \in H(\Omega)$.*

1. *(a_n) is bounded if and only if $(T^n f)$ is a normal family in $H(\Omega)$.*
2. *(a_n) tends to 0 if and only if $(T^n f)$ tends to 0 in $H(\Omega)$.*

Proof 1. If $(T^n f)$ is a normal family then, in particular, $a_n = (T^n f)(0)$ form a bounded sequence. Suppose that, conversely, there is c with $|a_n| \leq c$. Then

$$|R_n f|(z) \leq \frac{c|z|^n}{\big|1 - |z|\big|} \quad (z \in \mathbb{C} \setminus \mathbb{T})$$

easily implies the local boundedness of $(T^n f)$ on $\Omega \setminus \mathbb{T}$. The local boundedness near each point of $\Omega \cap \mathbb{T}$ is essentially the proof of the classical theorem of M. Riesz on boundedness of partial sums of Taylor series with bounded coefficients on arcs of holomorphy (see e.g. [27], p. 244). According to Montel's theorem, $(T^n f)$ is a normal family.

2. As above, the necessity of (a_n) being a zero sequence is clear. If (a_n) tends to zero then also $c_n := \max\{|a_{n+\nu}| : \nu \in \mathbb{N}_0\}$ tends to 0. Thus, for all $z \in \mathbb{D}$,

$$|T^n f|(z) \leq \frac{c_n}{1 - |z|} \to 0 \quad (n \to \infty).$$

Since $(T^n f)$ is a normal family, Vitali's theorem (see e.g. [27], p. 157) implies the convergence of $(T^n f)$ to 0 in $H(\Omega)$. □

For $h \in H(\Omega)$ we write $Z(h) := \{z \in \Omega : h(z) = 0\}$. As a consequence of Theorem 2 we obtain

Corollary 1 *Let Ω be a domain with $\mathbb{D} \subset \Omega$ and let $f \in H(\Omega)$ with bounded sequence of Taylor coefficients.*

1. *If $E \subset \mathbb{T}$ is a (Λ, α)-Dirichlet set then there exist a subsequence of $(R_n f)_{n \in \Lambda}$ and a function $h \in H(\Omega)$ with $R_n f \to \alpha h$ uniformly on E.*
2. *If $E \subset \Omega \cap \mathbb{T}$ is closed and if a subsequence $(R_{n_j} f)_j$ of $(R_n f)_n$ converges uniformly on E to some function g then there exist functions $h \in H(\Omega)$ and $\alpha \in C(E \setminus Z(h), \mathbb{T})$ with $g = \alpha h$. Moreover, in this case a subsequence of (ζ^{n_j}) tends to $\alpha(\zeta)$ locally uniformly on $E \setminus Z(h)$. In particular, $E \setminus Z(h)$ is a pseudo Dirichlet set.*

Proof 1. Since by Theorem 2 the familiy $(T^n f)_{n\in\Lambda}$ is a normal, a subsequence $(T^{n_j})_j$ of $(T^n)_{n\in\Lambda}$ converges to some function $h \in H(\Omega)$. Then, according to (1), the sequence $(R_{n_j} f)$ converges to αh uniformly on E.

2. Since $(T^{n_j} f)$ is a normal family, a subsequence $(T^{n_{j_k}} f)$ converges to some function $h \in H(\Omega)$. In the case $h = 0$ there is nothing more to prove. If h is not the zero function, then $Z(h) \cap E$ is finite (maybe empty). Then (1) implies that $\zeta^{n_{j_k}}$ tends to $\alpha(\zeta) := (g/h)(\zeta)$ for all $\zeta \in E \setminus Z(h)$, and the convergence is uniform on compact parts of $E \setminus Z(h)$. □

Remark 1 As Theorem 2 and (2) show, in the case of a sequence of Taylor coefficients tending to 0 the sequences $(T^n f)$ and $(R_n f)$ tend to $h = 0$ locally uniformly on $\Omega \cap \mathbb{T}$. We recover the classical Fatou-Riesz theorem saying that $(S_n f)$ tends to f uniformly on each closed arc of holomorphy of f (see e.g. [27, p. 244]).

If we do no longer restrict to Taylor series with bounded coefficients, more general limit functions may occur, as the next result shows. It is a slight strengthening of a result of Beise, Meyrath and Müller ([2, Theorem 2.2]).

Theorem 3 *Let Ω be a domain with $\mathbb{D} \subset \Omega$ such that each component of $\mathbb{C}_\infty \setminus \Omega$ meets $\mathbb{T}$. If $E \subset \Omega \cap \mathbb{T}$ is a Dirichlet set, then generically all functions in $H(\Omega)$ enjoy the following property: For each continuous function h on E there is a subsequence of $(R_n f)$ that converges to h uniformly on E and to 0 locally uniformly on $\Omega \cap \mathbb{T} \setminus E$.*

Proof Let E be a (Λ, α)-Dirichlet set and let $(B_j)_{j\in\mathbb{N}}$ be an exhausting sequence of closed subsets of $\Omega \cap \mathbb{T} \setminus E$. Then the compact set $E \cup B_j$ has connected complement with respect to $\mathbb{C}$. If we fix a point $a \in \mathbb{C}_\infty \setminus \Omega$, then, according to Mergelian's theorem, the rational functions with pole at a are dense in $C(E \cup B_j)$. Hence, for each $j \in \Lambda$ there is a rational function r_j with pole only at a and $\|r_j - h/\alpha\|_E < 1/j$ as well as $\|r_j\|_{B_j} < 1/j$.

Since each component of $\mathbb{C}_\infty \setminus \Omega$ meets $\mathbb{T}$, the operator T is topologically mixing (see [2], Theorem 1.1) which implies that generically all functions $f \in H(\Omega)$ are universal with respect to $(T^n)_{n\in\Lambda}$, that is, the set $\{T^n f : n \in \Lambda\}$ is dense in $H(\Omega)$. Let $f \in H(\Omega)$ be such a function and let $n_0 \in \Lambda$. For $j \in \mathbb{N}$ we choose $n_j \in \Lambda$ in such a way that $n_j > n_{j-1}$ and $\|T^{n_j} f - r_j\|_{E\cup B_j} < 1/j$. Then $(T^{n_j} f)_j$ tends to h/α uniformly on E and to 0 locally uniformly on $\Omega \cap \mathbb{T} \setminus E$. Since $(\zeta^{n_j})_j$ is a subsequence of $(\zeta^n)_{n\in\Lambda}$, the statement follows from (1) because E is a (Λ, α)-Dirichlet set. □

In the sequel, we say that a closed set $E \subset \mathbb{T}$ is a *set of universality* for a Banach space X of functions holomorphic in some domain Ω with $0 \in \Omega$ if generically all $f \in X$ have the property that for each $g \in C(E)$ a subsequence of $(S_n f)$ tends to g uniformly on E. Theorem 3 shows, in particular, that each Dirichlet set is a set of universality for $H(\Omega)$ if Ω is a domain with $\mathbb{D} \subset \Omega$ such that each component of $\mathbb{C}_\infty \setminus \Omega$ meets $\mathbb{T}$. A maximal domain to which this applies is the punctured extended plane $\mathbb{C}_\infty \setminus \{1\}$.

Remark 2 1. Let Ω be a bounded domain. For $p \geq 1$ the Bergman space $A^p(\Omega)$ is defined by

$$A^p(\Omega) := \{f \in H(\Omega) : \int_\Omega |f|^p \, d\lambda_2 < \infty\}.$$

Equipped with the norm $\|f\|_p := (\int_\Omega |f|^p \, d\lambda_2)^{1/p}$ for $f \in A^p(\Omega)$, the Bergman spaces become Banach spaces with $A^p(\Omega)$ densely and continuously embedded in $A^q(\Omega)$ for $p > q$. It is known that all functions in $A^p(\mathbb{D})$ satisfy

$$a_n = o(n^{1/p})$$

For the corresponding results we refer to [12].

In [3] it is shown that in the case of a Jordan domain Ω with $\mathbb{D} \subset \Omega$ and the property that $\mathbb{T} \setminus \Omega$ contains an arc, the Taylor shift T is mixing on $A^p(\Omega)$. With essentially the same proof as in the case of $H(\Omega)$, it follows that each Dirichlet set $E \subset \mathbb{T} \cap \Omega$ is a set of universality for $A^p(\Omega)$ and arbitrary $p \geq 1$ (see [3]). Here, also the subsequences can be chosen in such a way that in addition convergence to 0 holds locally uniform on $\Omega \cap \mathbb{T} \setminus E$. In particular, it turns out that the boundedness of the Taylor coefficients in the second statement of Corollary 1 cannot be weakened to a growth restriction of the form $a_n = o(n^\varepsilon)$ for any positive ϵ.

2. Let Ω be a domain with $\mathbb{D} \subset \Omega$ such that each component of $\mathbb{C}_\infty \setminus \Omega$ meets $\mathbb{T}$. Then it turns out that T is also mixing as a mapping on the space $H(\Omega) \oplus C(K)$, where $K := \mathbb{T} \setminus \Omega$ is the part of $\mathbb{T}$ that does not belong to Ω (see e.g. [2]). This implies that in the situation of Theorem 3 the statement concerning sets of universality can be extended to K: For generically all $f \in H(\Omega)$ it turns out that for each continuous function h on $E \cup K$ there is a subsequence of $(R_n f)$ that converges to h uniformly on E and to 0 locally uniformly on $\Omega \cap \mathbb{T} \setminus E$. A similar statement concerning universality on K does no longer hold in the case of $A^p(\Omega)$, as follows e.g. from [15, Corollary 2] or [29].

3 Spurious limit functions and Cesàro summability

A function $f \in H(\mathbb{D})$ is called Cesàro summable at $\zeta \in \mathbb{T}$ if the sequence $(\sigma_n f)$ of the arithmetic means

$$\sigma_n f := \frac{1}{n+1} \sum_{k=0}^{n} S_k f$$

converges at the point ζ. The classical Abel limit theorem (in an extended version) states that Cesàro-summability at ζ implies the existence of the non-tangental limit $f_\triangleright(\zeta)$ (see e.g. [31]). On the other hand, according to a theorem of Offord (see [25]), a function f is Cesàro summable at $\zeta \in \mathbb{T}$ if the radial limit of f at ζ exists and if, in addition,

$$a_n = o(n)$$

and f is bounded in $\mathbb{D} \cap \{z \in \mathbb{C} : |z - \zeta| < \delta\}$ for some positive δ. Note that Abel's and Offord's theorem imply that for functions f in the Hardy space H^∞ and $\zeta \in \mathbb{T}$ the following statements are equivalent:

- f has a nontangential limit at ζ,
- f has a radial limit at ζ,
- $(\sigma_n f)$ converges at ζ (and then to $f_\triangleright(\zeta)$).

More equivalent conditions are found in [14]. If f belongs to the disc algebra $A(\overline{\mathbb{D}})$, the conditions are satisfied at all points $\zeta \in \mathbb{T}$ (which is here also a consequence of the classical Fejér theorem on convergence of the partial sums of Fourier series in $C(\mathbb{T})$).

In view of Theorem 1, in a natural way the question raises to what extent limit functions may appear in case of Cesàro summability. It turns out that Cesàro summability actually imposes restrictions concerning the existence of spurious limit functions, at least in the case of uniform limits. In order to formulate a necessary condition, we recall the notion of porosity.

For $A \subset \mathbb{R}$ and $x \in \mathbb{R}$ let $p^+(A, x) := \limsup_{r \to 0} r^{-1} \lambda^+(A, x, r)$, where $\lambda^+(A, x, r)$ denotes the supremum of lengths of intervals lying in $(x, x+r) \setminus A$. The set A is called (upper) porous from the right at the point x if $p^+(A, x) > 0$. If $p^+(A, x) = 1$ then A is called strongly (upper) porous from the right at x (see e.g. [30, Chapter 8]). Similarly, porosity and strong porosity from the left can be defined in terms of $p^-(A, x)$, where $(x, x+r)$ is replaced by $(x-r, x)$. Note that also lower one sided porosity may be defined in a similar way with $\limsup$ replaced by $\liminf$. It turns out that this is a considerably stronger condition (cf. [24, Chapter 11]). To give an idea of the kind of "thinness" strong porosity imposes, we mention that the set $\{1/2^k : k \in \mathbb{N}\}$ is not strongly porous from the right at 0, while the set $\{1/k! : k \in \mathbb{N}\}$ is.

If $E \subset \mathbb{T}$ we say that E is two sided strongly porous at $\zeta \in \mathbb{T}$ if E is of the form $E = \{\zeta e^{\pi i \theta} : \theta \in A\}$ for some set $A \subset (-1, 1]$ that is strongly porous at 0 from the right and from the left. It can be shown that Dirichlet sets are two sided strongly porous at all points (see [4]). This implies, in particular, that two sided strongly porous sets may have Hausdorff dimension 1 (which is not possible in the case of lower porosity; cf. [24, Chapter 11]). It would be interesting to have an example of a two sided strongly porous set which is not Dirichlet.

Based on Rogosinksi summability it is proved in [4] that strong two sided porosity at all points of Cesàro summability is a necessary condition for the appearance of spurious uniform limit functions:

Lemma 1 (Bernal, Jung, Müller) *Consider $f \in H(\mathbb{D})$ to be Cesàro summable at $\zeta \in \mathbb{T}$ and E a closed set in $\mathbb{T}$. If E is not two sided strongly porous at ζ, then each uniform limit g of a subsequence of $(S_n f)$ satisfies $g(\zeta) = f_\triangleright(\zeta)$.*

Combining the lemma with Offord's theorem from [25] mentioned above, we obtain in particular

- Each set of universality $E \subset \mathbb{T}$ for the disc algebra $A(\overline{\mathbb{D}})$ is two sided strongly porous at all points.
- If Ω is a bounded domain in $\mathbb{C}$ with $\mathbb{D} \subset \Omega$ then each set of universality $E \subset \mathbb{T} \cap \Omega$ for the Bergman space $A^1(\Omega)$ is two sided strongly porous at all points (note that $a_n = o(n)$ for each $f \in A^1(\Omega)$).

In Remark 2 we have noted that, in the opposite direction, each Dirichlet set is a set of universality for the Bergman spaces $A^p(\Omega)$, where $p \geq 1$ is arbitrary and Ω is a Jordan domain. For the disc algebra $A(\overline{\mathbb{D}})$ corresponding questions seem to be more delicate. It was proved by Herzog and Kunstmann ([19]) that each finite set $E \subset \mathbb{T}$ is a set of universality for $A(\overline{\mathbb{D}})$ and Papachristodoulos and Papadimitrakis ([26]) have shown, among others, that the right limit function f can simultaneously appear outside sets of vanishing Hausdorff dimension. Moreover, generically all compact sets in the hyperspace of $\mathbb{T}$, that is, the space of compact nonempty subsets of $\mathbb{T}$ endowed with the Hausdorff metric, are sets of universality for the disc algebra (see [4]).

4 No spurious limit functions: The Dirichlet space

We write $m_2 := \lambda_2/\pi$ for the area measure normalised with respect to the unit disc. The Dirichlet space D consists of all functions in $H(\mathbb{D})$ having derivative in the Bergman space $A^2 := A^2(\mathbb{D})$, that is,

$$D := \{f \in H(\mathbb{D}) : \int_{\mathbb{D}} |f'|^2 \, dm_2 < \infty\}.$$

Excellent introductions to the Dirichlet space are the monography [13] as well as the expository article [28]. It is easily seen that $f \in H(\mathbb{D})$ belongs to D if and only if $\sum_{\nu=1}^{\infty} \nu |a_\nu|^2 < \infty$ and in this case

$$\sum_{\nu=1}^{\infty} \nu |a_\nu|^2 = \int_{\mathbb{D}} |f'|^2 \, dm_2 \, .$$

Equipped with the scalar product

$$\langle f, g \rangle := a_0 \overline{b_0} + \sum_{\nu=1}^{\infty} \nu a_\nu \overline{b_\nu} \qquad (f, g \in D),$$

where $b_n = a_n(g)$ denotes the n-th Taylor coefficient of g, the Dirichlet space becomes a Hilbert space with the monomials $p_k(z) := z^k$ as an orthogonal basis. We write $\| \cdot \|$ for the induced norm.[2] Then

$$\|f - S_n f\|^2 = \sum_{\nu=n+1}^{\infty} \nu |a_\nu|^2 \to 0 \qquad (n \to \infty)$$

[2] In [13] a slightly different but equivalent Hilbert space norm is chosen.

implies convergence of $(S_n f)$ to f in D. There is also a quite smart behaviour in respect to pointwise convergence. From Abel's theorem and a result of Fejér (see e.g. [23, p. 65]) we obtain that for $\zeta \in \mathbb{T}$ the following statements are equivalent:

- f has a nontangential limit at ζ,
- f has a radial limit at ζ,
- $(S_n f)$ converges at ζ (and then to $f_{\triangleright}(\zeta)$).

Thus, in D spurious limit functions do not appear at all.[3]

According to Beurling's theorem ([5], [13, Theorem 3.2.1]), each function in D has nontangential limits, not only almost everywhere but quasi everywhere, that is, except for a polar set. Actually, due to a result of Twomey this holds even for oricyclic limits. Conversely by a result of Carleson ([13, Theorem 3.4.1], [28, Theorem 5.4]), it is known that for each closed polar set $E \subset \mathbb{T}$ there are functions in D which do not have a radial limit at any point of E. For further information concerning the boundary behaviour of functions in D we refer to [13, Chapter 3] and [28, Section 5].

It turns out that a strong form of divergence of $(S_n f)$ on closed polar sets E generically takes place in D:

Theorem 4 *Each closed and polar set $E \subset \mathbb{T}$ is a set of universality for D.*

As formulated in [10, Lemma 2.5] (cf. also the proof of Theorem 1.1 in [3]), an application of the universality criterion (see [17] or [18]) shows that it suffices to prove the following result on simultaneous approximation in D and $C(E)$ by polynomials.

Theorem 5 *If $E \subset \mathbb{T}$ is a closed polar set then for all $(f, g) \in D \times C(E)$ and all $\varepsilon > 0$ there is a polynomial p such that $\|f - p\| < \varepsilon$ and $\|g - p\|_E < \varepsilon$.*

Proof Being a Hilbert space, D equals its norm dual D^* by identifying g and $\langle \cdot, g \rangle$. Moreover, the norm dual of $C(E)$ is the space of Borel measures supported on E (with the total variation norm). Using the Hahn-Banach theorem, the statement on simultaneous approximation can be transformed into an equivalent one saying that no non-zero Cauchy transform

$$\widehat{g}(z) := \sum_{\nu=0}^{\infty} \langle p_\nu, g \rangle z^\nu = \overline{b_0} + \sum_{\nu=1}^{\infty} \nu \overline{b_\nu} z^\nu \quad (z \in \mathbb{D}) \tag{3}$$

of a function $g \in D$ can coincide (on $\mathbb{D}$) with the Cauchy transform

$$\widehat{\mu}(z) := \int \frac{1}{1 - z\overline{\zeta}} \, d\mu(\zeta) \qquad (z \in \mathbb{C} \setminus E)$$

[3] This also implies that for functions $f \in D \cap A(\overline{\mathbb{D}})$ the partial sums $S_n f$ always converge to f on $\mathbb{T}$, even uniformly according to Fejér's result mentinoned above. From Carathéodory's theorem it follows that the conformal mappings from the unit disc $\mathbb{D}$ to domains of bounded area and having locally connected complement form a subclass of $D \cap A(\overline{\mathbb{D}})$.

of a complex Borel measure with support in E (cf. [3, Lemma 2.1] or [10, Lemma 2.7]).

In order to prove that this is true, let us suppose to have $\widehat{g} = \widehat{\mu}$ on $\mathbb{D}$. Since $h \in H(\mathbb{D})$ belongs to A^2 if and only if $\sum_{\nu=1}^{\infty} |a_\nu(h)|^2/\nu < \infty$, from (3) it is seen that $\widehat{g} \in A^2$. Since μ has support in E and $\widehat{\mu}$ vanishes at ∞, we have $\widehat{\mu} \in H(\mathbb{C}_\infty \setminus E)$. Tumarkin's theorem (see e.g. [7, Theorem 5.3.1]) shows that

$$\sup_{0<r<1} \int_{\mathbb{T}} |\widehat{\mu}(r\zeta) - \widehat{\mu}(\zeta/r)|\, dm(\zeta) < \infty.$$

The function $\varphi(z) := \widehat{\mu}(z) - \widehat{\mu}(1/\overline{z})$ is harmonic in $\mathbb{D}$ with Poisson integral

$$\varphi(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|w-z|^2}\, d\mu(w) \quad (z \in \mathbb{D}).$$

A variant of the Hardy-Littlewood inequality (see [11, Theorem 5.9]) using the Poisson integral instead of the Cauchy integral shows that

$$\int_{\mathbb{T}} |\varphi(r\zeta)|^2\, dm(\zeta) = O((1-r)^{-1/2}) \quad (r \to 1^-),$$

and integration along r implies $\varphi \in L^2(\mathbb{D}, m_2)$. Since

$$\widehat{\mu}(1/\overline{z}) = \widehat{\mu}(z) - \varphi(z) \quad (z \in \mathbb{D})$$

and $\widehat{\mu} \in A^2$ we obtain that also the (harmonic) function on the left hand side belongs to $L^2(\mathbb{D}, m_2)$. But then $\widehat{\mu}$ is square m_2-integrable in the annulus $2\mathbb{D} \setminus \overline{\mathbb{D}}$ and therefore belongs to $A^2(2\mathbb{D} \setminus E)$. Since compact polar sets are sets of removable singularities for Bergman spaces $A^2(\Omega)$ (see e.g. Theorem 9.5 in [8]), each function in $A^2(2\mathbb{D} \setminus E)$ extends holomorphically to $2\mathbb{D}$. Hence, $\widehat{\mu}$ extends holomorphically to the extended plane (and vanishes at ∞). Thus, we arrive at $\widehat{g} = \widehat{\mu} = 0$ on $\mathbb{D}$. □

Acknowledgements The author acknowledges the contribution of the anonymous referee whose very careful reading of the manuscript helped to improve the presentation.

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