Spherical Universality of Composition Operators

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Let Ω be an open subset of the complex plane and let ϕ be an injective holomorphic self-map of Ω such that the sequence of iterates of ϕ is a run-away sequence. We prove that the composition operator C_ϕ with symbol ϕ is spherically universal on a suitable function space consisting of sphere-valued functions – in contrast to the known fact that, in general, C_ϕ is not hypercyclic on $H(\Omega)$ in case that Ω is multiply connected. Moreover, concrete open sets which support spherically universal functions will explicitly be determined in case that the symbol of the composition operator is given by a finite Blaschke product of degree two that has an attracting fixed point at the origin.

1. Introduction

Given topological spaces X, Y and a family $\{T_\iota : \iota \in I\}$ of continuous mappings $T_\iota : X \to Y$, an element $x \in X$ is called *universal* for $\{T_\iota : \iota \in I\}$ if the set $\{T_\iota(x) : \iota \in I\}$ is dense in Y. In case that $T : X \to X$ is continuous and $T^n := T \circ \ldots \circ T$ denotes the n-th iterate of T, an element $x \in X$ is called *universal* for T if it is universal for the family $\{T^n : n \in \mathbb{N}\}$, i.e. if its orbit $\{T^n(x) : n \in \mathbb{N}\}$ is dense in X. We say that a property is fulfilled by *comeagre many* elements of a Baire space X if it is fulfilled on a comeagre subset of the corresponding space, i.e. this set contains a dense G_δ-set in X. According to the Birkhoff transitivity theorem (see e.g. [10, Theorem 1.16]), in the case of a Polish space X a universal element for T exists if and only if T is topologically transitive on X and in this case comeagre many elements turn out to be universal.

An intensively investigated class of operators consists of the class of composition operators on spaces of holomorphic functions on open subsets of the complex plane; see e.g. [1] and [10]. For an open subset Ω of the complex plane $\mathbb{C}$ and a holomorphic self-map ϕ of Ω, the *composition operator* with *symbol* ϕ is defined by

$$C_\phi : H(\Omega) \to H(\Omega), \ \ C_\phi(f) := f \circ \phi,$$

where $H(\Omega)$ denotes the Fréchet space of functions holomorphic in Ω endowed with the topology of locally uniform convergence. In the sequel, we also write $\phi^n := \phi \circ \ldots \circ \phi$ for the n-th iterate of ϕ and $M^* := M \setminus \{0\}$ for sets $M \subset \mathbb{C}$.

For fixed $0 < |\lambda| < 1$, let ϕ denote the automorphism $\phi : \mathbb{C}^* \to \mathbb{C}^*$, $\phi(z) := \lambda z$. In [3] (see also [9] and [4]) it was proved that the set of all functions $f \in H(\mathbb{C}^*)$ which have the property that $\{(C_\phi)^n(f)|_U : n \in \mathbb{N}\} = \{f \circ \phi^n|_U : n \in \mathbb{N}\}$ is dense in $H(U)$ for all simply connected open sets $U \subset \mathbb{C}^*$ is a comeagre set in $H(\mathbb{C}^*)$. Roughly speaking, this result relies on the following three reasons: Firstly, the symbol ϕ is injective, secondly, the sequence of iterates (ϕ^n) fulfils the "run-away-behaviour" $\phi^n \to 0 \in \partial\mathbb{C}^*$ locally uniformly, and finally, only simply connected open subsets U of the punctured complex plane are considered. The last restriction is necessary because it is easily seen that there does not exist a single function $f \in H(\mathbb{C}^*)$ which is universal for C_ϕ – in particular, the operator C_ϕ is not topologically transitive on $H(\mathbb{C}^*)$ (cf. [3, Remark p.55], see also [10, Corollary 4.30, Proposition 4.31]). Indeed, assuming that such a function f exists, we could find a strictly increasing sequence (n_k) in $\mathbb{N}$ such that $(f \circ \phi^{n_k})$ converges to 0 uniformly on the unit circle $\partial\mathbb{D}$. Hence, for $M := \max_{w \in \partial\mathbb{D}} |f(w)|$, there would exist some $N \in \mathbb{N}$ such that for all $k \geq N$ and all $w \in \partial\mathbb{D}$ we had $|(f \circ \phi^{n_k})(w)| \leq M$. Thus, for all $k \geq N$ and all $|z| = |\lambda|^{n_k}$, we would obtain $|f(z)| = |(f \circ \phi^{n_k})(z/\lambda^{n_k})| \leq M$. Therefore, the maximum modulus principle would imply $|f(z)| \leq M$ for all $|\lambda|^{n_k} \leq |z| \leq 1$ and all $k \geq N$, and we would obtain $|f| \leq M$ on $\mathbb{D}^*$. But this clearly contradicts the denseness of $\{f \circ \phi^n : n \in \mathbb{N}\}$ in $H(\mathbb{C}^*)$.

The fact that, in the above situation, the maximum modulus principle does not allow universality of C_ϕ on the whole punctured complex plane, now leads to the idea of considering composition operators between spaces of meromorphic functions. The main aim of this work is to show that the situation changes in an essential way if we consider compositional universality in the spherical metric. This will be done in Section 2, where firstly a general result on compositional universality in the spherical setting will be proved. Subsequently, this result will be applied to the case of holomorphic symbols having an attracting fixed point. Finally, if the symbol is given by a finite Blaschke product B of degree two on the unit disk that has an attracting fixed point at the origin, concrete open sets which allow universal functions for C_B will explicitly be determined in Section 3.

2.   Compositional Universality in $M_\infty(\Omega)$

For an open set $\Omega \subset \mathbb{C}$, we say that a function f which maps Ω to the extended complex plane $\mathbb{C}_\infty$ is *spherically meromorphic* if each restriction of f to a connected component of Ω is either meromorphic or else constantly infinity. We write $M_\infty(\Omega)$ for the set of spherically meromorphic functions on Ω. Then $M_\infty(\Omega)$ endowed with the topology of spherically locally uniform convergence turns out to be a completely metrizable space (cf. [6, Chapter VII]). A metric on $M_\infty(\Omega)$ inducing its topology is given by

$$\rho(f, g) := \sum_{n \in \mathbb{N}} 2^{-n} \frac{\rho_n(f, g)}{1 + \rho_n(f, g)},$$

where $\rho_n(f, g) := \max_{z \in K_n} \chi(f(z), g(z))$ with χ denoting the chordal distance on $\mathbb{C}_\infty$ and (K_n) being a compact exhaustion of Ω. A basis of the topology of spherically locally uniform convergence is given by $\{V_{\varepsilon, K, \Omega}(f) : \varepsilon > 0, K \subset \Omega \text{ compact}, f \in$

$M_\infty(\Omega)\}$, where

$$V_{\varepsilon,K,\Omega}(f) := \big\{ g \in M_\infty(\Omega) : \max_{z \in K} \chi(g(z), f(z)) < \varepsilon \big\}.$$

For a holomorphic self-map ϕ of Ω we consider the composition operator

$$C_\phi : M_\infty(\Omega) \to M_\infty(\Omega), \ \ C_\phi(f) := f \circ \phi.$$

Then C_ϕ is continuous with $(C_\phi)^n = C_{\phi^n}$ for all $n \in \mathbb{N}$.

Studying the proofs of the main results in [3] and [9], one can recognize the following rough outline of how to prove universality properties of a composition operator: It is assumed (and even necessary) that the symbol is injective and that the sequence of iterates of the symbol shows some kind of run-away behaviour (cf. the statement of Theorem 2.3 below). Subsequently, a suitable function is constructed in such a way that it can be approximated uniformly by rational functions having poles only outside a given compact set. This approximation, which is the crucial step of the proofs in [3] and [9], is obtained by an application of Runge's theorem on rational approximation of holomorphic functions. In order to obtain universality in the meromorphic setting, we need the following variant of Runge's theorem (see e.g. [8, Satz 11.1]):

THEOREM 2.1 (Spherical Runge Theorem) *Let $K \subset \mathbb{C}$ be compact, $U \supset K$ open, $f \in M(U)$ and $\varepsilon > 0$. Then there exists a rational function R such that $f - R$ is holomorphic on an open neighbourhood of K and such that*

$$\max_{z \in K} |f(z) - R(z)| < \varepsilon.$$

The spherical Runge theorem is a straightforward application of the classical version (indeed, in order to prove this statement, one can choose an open neighbourhood $U' \subset U$ of K on which f has only finitely many poles $z_1, \dots, z_N$ so that one can apply the classical Runge theorem to the holomorphic function $f - \sum_{n=1}^{N} h_n$, where h_n is the principal part of the Laurent series expansion of f at z_n). As an immediate consequence we obtain that for arbitrary open sets Ω in $\mathbb{C}$ the rational functions (restricted to Ω) are dense in $M_\infty(\Omega)$. In particular, the spherical Runge theorem implies that $M_\infty(\Omega)$ is separable and thus a Polish space.

We now fix an open set $\Omega \subset \mathbb{C}$, an open set $\Omega_0 \subset \mathbb{C}$ with $\Omega_0 \supset \Omega$ and a holomorphic function $\phi : \Omega \to \Omega$, and we write $\partial_\infty \Omega_0$ for the boundary of Ω_0 with respect to $(\mathbb{C}_\infty, \chi)$.

Definition 2.2 For $U \subset \Omega$ open, a function $f \in M_\infty(\Omega_0)$ is called $M_\infty(U)$-*universal* for C_ϕ if the set $\{f \circ \phi^n|_U : n \in \mathbb{N}\}$ is dense in $M_\infty(U)$, that is, f is universal for the sequence $(C_{\phi^n, U})$ of composition operators

$$C_{\phi^n, U} : M_\infty(\Omega_0) \to M_\infty(U), \ C_{\phi^n, U}(f) := f \circ \phi^n|_U.$$

Similarly to the holomorphic setting (cf. [12, Theorem 2.2]), the following universality result holds:

THEOREM 2.3   *Let $U \subset \Omega$ be open such that $\phi^n|_U$ is injective for each $n \in \mathbb{N}$ and such that the sequence $(\operatorname{dist}(\phi^n(\cdot), \partial_\infty \Omega_0))_{n \in \mathbb{N}}$ converges to 0 locally uniformly on U. Then comeagre many functions in $M_\infty(\Omega_0)$ are $M_\infty(U)$-universal for C_ϕ.*

Proof: According to the universality criterion (see e.g. [10, Theorem 1.57]), it suffices to show that the sequence $(C_{\phi^n, U})$ is topologically transitive.

In order to do so, let $f \in M_\infty(\Omega_0)$, $g \in M_\infty(U)$ as well as $K \subset \Omega_0$, $L \subset U$ compact and $\varepsilon > 0$ be given. Due to the assumption, we have uniform convergence $\phi^n|_L \to \partial_\infty \Omega_0$. Thus, setting $\delta := \operatorname{dist}(K, \partial_\infty \Omega_0)$, there exists an $N \in \mathbb{N}$ with $\operatorname{dist}(\phi^N(z), \partial_\infty \Omega_0) < \delta$ for all $z \in L$, which implies $K \cap \phi^N(L) = \emptyset$. Moreover, by assumption, the restriction $\phi^N|_U$ is injective so that the function

$$\varphi : K \cup \phi^N(L) \to \mathbb{C}_\infty, \quad \varphi(z) := \begin{cases} f(z), & \text{if } z \in K \\ g\big((\phi^N|_U)^{-1}(z)\big), & \text{if } z \in \phi^N(L) \end{cases}$$

is well-defined. As the disjoint sets K and $\phi^N(L)$ are compact with $K \subset \Omega_0$ and $\phi^N(L) \subset \phi^N(U)$ and since we have $f \in M_\infty(\Omega_0)$ as well as $g \in M_\infty(U)$, we see that φ can be extended spherically meromorphic to an open neighbourhood of $K \cup \phi^N(L)$. Hence, the spherical Runge theorem yields a rational function R such that

$$\max_{z \in K \cup \phi^N}(L) \chi(\varphi(z), R(z)) < \varepsilon.$$

In particular, due to $\varphi = f$ on K we obtain

$$\max_{z \in K} \chi(f(z), R(z)) = \max_{z \in K} \chi(\varphi(z), R(z)) \leq \max_{z \in K \cup \phi^N(L)} \chi(\varphi(z), R(z)) < \varepsilon$$

and thus $R|_{\Omega_0} \in V_{\varepsilon, K, \Omega_0}(f)$. Because of $\varphi = g \circ (\phi^N|_U)^{-1}$ on $\phi^N(L)$, we further obtain

$$\max_{z \in L} \chi\big(g(z), C_{\phi^N, U}(R|_{\Omega_0})(z)\big) = \max_{z \in L} \chi\big(g(z), R(\phi^N(z))\big)$$
$$= \max_{w \in \phi^N(L)} \chi\big(g((\phi^N|_U)^{-1}(w)), R(w)\big) = \max_{w \in \phi^N(L)} \chi(\varphi(w), R(w))$$
$$\leq \max_{w \in K \cup \phi^N(L)} \chi(\varphi(w), R(w)) < \varepsilon$$

and thus $C_{\phi^N, U}(R|_{\Omega_0}) \in V_{\varepsilon, L, U}(g)$. Altogether, it follows that $C_{\phi^N, U}(R|_{\Omega_0})$ is contained in $C_{\phi^N, U}(V_{\varepsilon, K, \Omega_0}(f)) \cap V_{\varepsilon, L, U}(g)$. This shows the topological transitivity of the sequence $(C_{\phi^n, U})$. $\qquad\square$

Remark 2.4   1. The proof of Theorem 2.3 runs similarly as the proof of the corresponding theorem in the holomorphic setting (cf. the proof of Theorem 2.2 in [12]). However, the crucial difference in the meromorphic setting lies in the fact that now the function which is yielded by Runge's theorem automatically belongs to the considered function space (that is $R|_{\Omega_0} \in M_\infty(\Omega_0)$) – whereas, in the holomorphic setting, we do not have $R|_{\Omega_0} \in H(\Omega_0)$ in general because of possible poles of R in Ω_0. Then, a sufficient condition under which it is guaranteed that R can be chosen to have no poles in Ω_0 is given if the set U is simply connected (cf. the proof of Theorem 2.2 in [12]).

2. The proof of Theorem 2.3 shows that the sequence $(C_{\phi^n, U})$ is actually topologically mixing (for a definition, see e.g. [10, Definition 1.56]).

In the sequel we restrict ourselves to the case $\Omega_0 = \mathbb{C}^*$, that is, we ask for the existence of universal functions which are meromorphic in $\mathbb{C}^*$.

The first application of Theorem 2.3 deals with the case of a symbol ϕ which is holomorphic near an attracting fixed point, which we suppose to be 0, i.e. we have $\phi(0) = 0$ and $0 < |\lambda| < 1$ for $\lambda := \phi'(0)$. According to G. Kœnigs' linearization theorem, there exist open neighbourhoods U and V of 0 as well as a conformal mapping $\varphi : U \to V$ which conjugates the map $\phi|_U : U \to U$ to the linear function $\ell : V \to V$, $\ell(w) := \lambda w$, i.e.

$$\varphi \circ \phi^n = \ell^n \circ \varphi = \lambda^n \cdot \varphi$$

holds on U for all $n \in \mathbb{N}$ (see e.g. [5, Theorem II.2.1]). In particular, we obtain $\varphi(0) = 0$ and locally uniform convergence $\phi^n \to 0$ on U. Since ℓ, φ and φ^{-1} are injective, the same is also true for $\phi|_U = \varphi^{-1} \circ \ell \circ \varphi$. Hence, the restriction $\phi|_{U^*} : U^* \to U^*$ is injective and Theorem 2.3 implies

COROLLARY 2.5   *Comeagre many functions in $M_\infty(\mathbb{C}^*)$ are $M_\infty(U^*)$-universal for C_ϕ.*

Remark 2.6   In the special case $V = U = \mathbb{C}$ and $\phi = \ell$, we obtain – in sharp contrast to the same situation in the holomorphic setting (cf. the example in Section 1) – that the operator C_ϕ is topologically transitive on $M_\infty(\mathbb{C}^*)$ (and even topologically mixing; cf. Remark 2.4)

It is easily seen that whenever we consider an invariant open set U with $0 \notin U$ and so that ϕ is not injective on U, there cannot exist a function $f \in M_\infty(U)$ which is $M_\infty(U)$-universal for C_ϕ, that is $C_\phi : M_\infty(U) \to M_\infty(U)$ cannot be universal. In particular, this is the case for punctured open neighbourhoods of superattracting fixed points of the symbol – however, at least in the holomorphic setting, then C_ϕ fulfils a certain universality property on "many small" compact subsets of such a neighbourhood (cf. [12], Theorem 3.5).

Let now ϕ be a holomorphic self-map on an open set Ω and let

$$A := A_{\phi,0} := \{z \in \Omega : \phi^n(z) \to 0\}$$

denote the basin of attraction of 0 under ϕ. It is easily seen that A is completely invariant under ϕ and open, this is, $\phi \in H(A)$ and $\phi(A) \subset A$ as well as $\phi^{-1}(A) \subset A$ (see e.g. [5, p. 28]). It is well-known that the function φ from above can be extended to a holomorphic function Φ on the whole basin of attraction A and that the equation $\Phi \circ \phi = \lambda \cdot \Phi$ still holds on A (see e.g. [5, p. 32]). Moreover, it can be shown that this property determines the function Φ up to a multiplicative nonzero constant (cf. [5, p. 28]). We consider the backward orbit

$$O^- := O^-_{\phi,0} := \bigcup_{n \in \mathbb{N}} \{z \in \Omega : \phi^n(z) = 0\}$$

of 0 under ϕ and we write $A^- := A \setminus O^-$. Again, A^- is completely invariant under

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ϕ and open but ϕ is in general not injective on A^-. Our aim is to determine the set of limit functions of the sequence $(f \circ \phi^n|_{A^-})$ for generic functions $f \in M_\infty(\mathbb{C}^*)$. For $U \subset A^-$ open we write $\omega(U, f, \phi)$ for the set of all spherically locally uniform limit functions of the sequence $(f \circ \phi^n|_U)$.

THEOREM 2.7   *Let $\phi \in H(\Omega)$ be a self-map and so that $\bigcap_{n \in \mathbb{N}} \phi^n(\Omega)$ is a neighbourhood of 0. If $f \in M_\infty(A^-)$ is $M_\infty(U^*)$-universal for some open neighbourhood $U \subset \bigcap_{n \in \mathbb{N}} \phi^n(\Omega)$ of 0 on which ϕ is conjugated to $w \mapsto \lambda w$ then*

$$\omega(A^-, f, \phi) = \{g \circ \Phi|_{A^-} : g \in M_\infty(\Phi(A^-))\}.$$

Proof: Combining the statements of Corollary 4.4, Lemma 4.5 and Lemma 4.6 in [12], the proof runs exactly in the same way as the proof of the corresponding result in the holomorphic setting (cf. [12, Theorem 4.7]). One only has to observe the following three points: Due to the assumption that U is contained in the 0-neighbourhood $\bigcap_{n \in \mathbb{N}} \phi^n(\Omega)$, one can prove analogously to the proof of Lemma 4.3 in [12] (which is need for the proof of Corollary 4.4) that we have

$$\omega\big(\phi^{-n}(U), f, \phi\big) \supset \big\{g \circ \Phi|_{\phi^{-n}(U)} : g \in M_\infty(\Phi(\phi^{-n}(U)))\big\}, \;\; n \in \mathbb{N}_0.$$

Moreover, in view of Corollary 2.5, it now suffices to consider the punctured neighbourhood U^* instead of the set U_0 which was constructed in the proof of Lemma 4.5. ii) in [12]. Finally, an analogous statement as that of Lemma 4.5 in [12] now also holds in the meromorphic setting. $\qquad\square$

Corollary 2.5 shows that a neighbourhood U of 0 exists so that comeagre many $f \in M_\infty(\mathbb{C}^*)$ turn out to be $M_\infty(U^*)$-universal for C_ϕ. From Theorem 2.7 we obtain

COROLLARY 2.8   *Under the assumption of Theorem 2.7, comeagre many $f \in M_\infty(\mathbb{C}^*)$ enjoy the property that*

$$\omega(A^-, f, \phi) = \{g \circ \Phi|_{A^-} : g \in M_\infty(\Phi(A^-))\}.$$

Remark 2.9   The assumption of $\bigcap_{n \in \mathbb{N}} \phi^n(\Omega)$ being a neighbourhood of 0 is obviously satisfied in the case that $0 \in \Omega = \phi(\Omega)$. Moreover, according to Picard's theorem it is also satisfied for entire functions ϕ that do not omit 0 (with $\Omega = \mathbb{C}$).

Remark 2.10   For entire functions ϕ, Theorem 2.7 yields an analogous statement in the spherical setting as Theorem 4.7 in [12] does in the holomorphic setting. The main difference lies in the fact that we can make a statement about the generic set of limit functions on the whole basin of attraction of 0 under ϕ except the minimal (countable) set O^-. In contrast, the exceptional sets in the holomorphic setting are larger (unions of rectifiable curves with Hausdorff dimension 1) and cannot chosen to be minimal.

3.   Determination of sets of universality

Theorem 2.7 implies that $M_\infty(U)$-universal functions $f \in M_\infty(\mathbb{C}^*)$ exist for all open subsets U of A^- with the property that $\Phi|_U$ is injective. However, in general, we do not have any information on the shape or the exact location of such sets. In

this section, it is our aim to explicitly determine open sets $U \subset A^-$ which support $M_\infty(U)$-universal functions for C_ϕ in $M_\infty(\mathbb{C}^*)$ in case that the symbol is a rational function of a special form.

In order to do so, we need some information about ϕ on a larger neighbourhood of the attracting fixed point at the origin. Again, the concept of conformal conjugation will be the main consideration here. Apparently, ϕ should now be conjugated to a "simple" function on a "large" open set containing 0. This can be achieved as follows (cf. [7, p. 586]): For a rational function ϕ of degree $d_\phi \geq 2$, the *Fatou set* F_ϕ of ϕ is defined as the set of all points $z \in \mathbb{C}_\infty$ for which there exists a neighbourhood U of z such that $\{\phi^n|_U : n \in \mathbb{N}\}$ is a normal family in $M_\infty(U)$. Considering a simply connected invariant component G of F_ϕ with $G \subset \mathbb{C}$, $G \neq \mathbb{C}$, the Riemann mapping theorem implies the existence of a conformal map $\psi : G \to \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Hence, ψ conjugates $\phi|_G : G \to G$ to the function $f := \psi \circ \phi \circ \psi^{-1} : \mathbb{D} \to \mathbb{D}$. Since rational functions map components of their Fatou sets properly onto each other (see e.g. [16, Theorem 1 on p. 39]), we obtain that f is a proper self-map of $\mathbb{D}$ (i.e. $f^{-1}(K)$ is compact for each compact set $K \subset \mathbb{D}$). It is well-known that each proper self-map of $\mathbb{D}$ is the restriction to $\mathbb{D}$ of a finite Blaschke product (see e.g. [16, Exercise 6 on p. 7], or [15, p. 185]). The simplest subclass of finite Blaschke products having an attracting fixed point at the origin is given by the functions

$$B_\alpha : \mathbb{C}_\infty \to \mathbb{C}_\infty, \ \ B_\alpha(z) := z \cdot \varphi_\alpha(z) = z \cdot \frac{z - \alpha}{1 - \overline{\alpha}z}, \ \ \alpha \in \mathbb{D}^*.$$

From now on, we fix some $\alpha \in \mathbb{D}^*$ and we write $B := B_\alpha$. Calculating $B'(z) = 0$, we see that there exist exactly two critical points of B which are given by

$$z_1 := \frac{1 - \sqrt{1 - |\alpha|^2}}{\overline{\alpha}} \qquad \text{and} \qquad z_2 := \frac{1 + \sqrt{1 - |\alpha|^2}}{\overline{\alpha}}.$$

Another simple calculation yields $0 < |z_1| < |\alpha| < 1 < 1/|\overline{\alpha}| < |z_2|$ so that z_1 is the only critical point of B in $\mathbb{D}$. Because of $\arg \alpha = \arg(1/\overline{\alpha})$, the points 0, z_1, α, $1/\overline{\alpha}$ and z_2 lie on the same straight line through 0. Moreover, the two distances $|z_1 - 1/\overline{\alpha}|$ and $|z_2 - 1/\overline{\alpha}|$ are equal. For $z, w \in \mathbb{C}_\infty$, a short computation shows that we have $B(z) = B(w)$ if and only if $w = z$ or $w = -\varphi_\alpha(z)$. For $z \in \mathbb{C}_\infty$, we compute that we have $-\varphi_\alpha(z) = z$ if and only if $z = z_1$ or $z = z_2$. Thus, the critical points of B are exactly the fixed points of $-\varphi_\alpha$. Indeed, the map $-\varphi_\alpha$ is a key factor in understanding the dynamics of B because its geometry on $\mathbb{D}$ is known.

In order to describe this geometry, we introduce the following notations: For a straight line L in the complex plane, we denote by R_L the *reflection* map which reflects each point in $\mathbb{C}$ with respect to L. Considering a point $w \in \mathbb{C}$, a radius $r > 0$ and the closed disk $C := \{z \in \mathbb{C} : |z - w| \leq r\}$, we define $I_C(z)$ for $z \in \mathbb{C}\backslash\{w\}$ as the point which lies on the ray $\{w + t(z - w) : t \geq 0\}$ and which has distance $r^2/|z - w|$ to w. The map I_C is called the *inversion* on C. We have $I_C(C^\circ\backslash\{w\}) = \mathbb{C}\backslash C$ and $I_C(\mathbb{C}\backslash C) = C^\circ\backslash\{w\}$. Each point on ∂C is a fixed point of I_C. If $w = 0$ and $r = 1$, we have $I_{\overline{\mathbb{D}}}(z) = 1/\overline{z}$ for all $z \in \mathbb{C}^*$. For a closed disk D which is orthogonal to C (i.e. the two tangents to D through the two points in $\partial C \cap \partial D$ pass through w), it follows $I_C(D) = D$ (see [14, p. 149]).

Considering the finite Blaschke product $B = B_\alpha$, now let L be the straight line which passes through 0 and α. As stated above, we have z_1, z_2, $1/\overline{\alpha} \in L$ and

$0 < |z_1| < |\alpha| < 1 < 1/|\overline{\alpha}| < |z_2|$ as well as $|z_1 - 1/\overline{\alpha}| = |z_2 - 1/\overline{\alpha}|$ (see Figure 1). Moreover, the two critical points z_1 and z_2 of B are exactly the two fixed points of $-\varphi_\alpha$. Therefore, there exists a closed disk C with center $1/\overline{\alpha}$ which is orthogonal to $\overline{\mathbb{D}}$ with $I_C(\alpha) = 0$ and $\partial C \cap L = \{z_1, z_2\}$ and such that $-\varphi_\alpha$ acts on $\mathbb{D}$ as the composition of the reflection R_L and the inversion I_C in any order, i.e. we have

$$-\varphi_\alpha|_{\mathbb{D}} = (R_L \circ I_C)|_{\mathbb{D}} = (I_C \circ R_L)|_{\mathbb{D}} \tag{1}$$
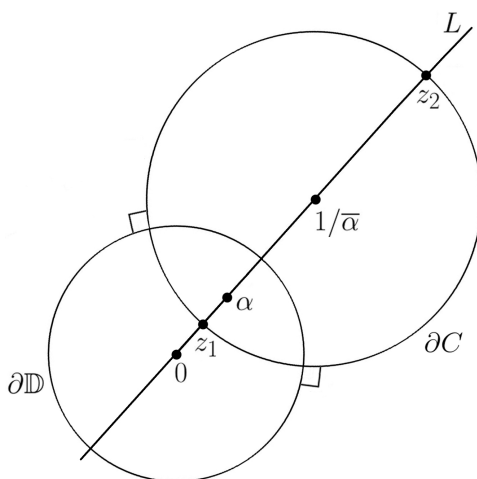
(see [14, p. 207]).

Figure 1.

We define the disjoint subsets S_1, S_2, S_3 and S_4 of $\mathbb{D}$ as illustrated in Figure 2. According to (1), we see that $-\varphi_\alpha$ interchanges S_1 and S_3 as well as S_2 and S_4,
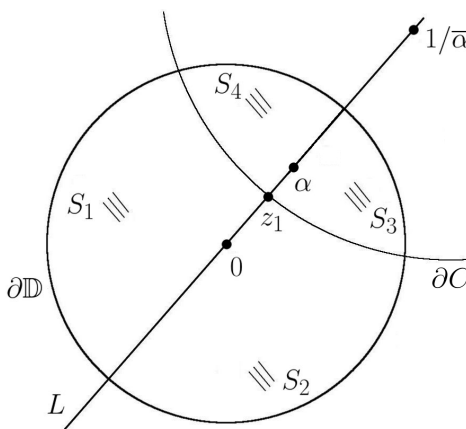

Figure 2.

i.e. we have $-\varphi_\alpha(S_1) = S_3$, $-\varphi_\alpha(S_3) = S_1$, $-\varphi_\alpha(S_2) = S_4$ and $-\varphi_\alpha(S_4) = S_2$.

LEMMA 3.1 *The finite Blaschke product B is injective on the unions $S_k \cup S_l$ for all $k, l \in \{1, 2, 3, 4\}$ with $\{k, l\} \notin \{\{1, 3\}, \{2, 4\}\}$.*

Proof: Let $z, w \in S_1 \cup S_2$ with $B(z) = B(w)$. Due to the above considerations, it

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follows that $z = w$ or $-\varphi_\alpha(z) = w$. But the latter cannot be true because in this case we would obtain

$$w = -\varphi_\alpha(z) \in -\varphi_\alpha(S_1 \cup S_2) = -\varphi_\alpha(S_1) \cup -\varphi_\alpha(S_2) = S_3 \cup S_4,$$

a contradiction. Hence, B is injective on $S_1 \cup S_2$. According to the geometry of $-\varphi_\alpha$, the injectivity of B on the other unions can be shown similarly. $\qquad\square$

In order to exemplify the universality statement of Corollary 2.5 in case of the finite Blaschke product B, we now have a look at the contour lines of B. For $0 < r < 1$ and $z \in \mathbb{D}$, we have $|B(z)| = r$ if and only if

$$|z| \cdot |z - \alpha| = r \cdot |1 - \overline{\alpha}z|. \tag{2}$$

Ignoring the factor $|1 - \overline{\alpha}z|$ on the right-hand side, the points $z \in \mathbb{C}$ fulfilling the equation $|z| \cdot |z - \alpha| = r$ would form a *Cassini oval*. In general, for two points $w_1, w_2 \in \mathbb{C}$ and a constant $c > 0$, the Cassini oval $C(w_1, w_2, c)$ is defined as the set of all points in the complex plane having the property that the product of their distances to w_1 and w_2 has constant value c^2, i.e.

$$C(w_1, w_2, c) := \big\{ z \in \mathbb{C} : |z - w_1| \cdot |z - w_2| = c^2 \big\}.$$

The shapes of these sets according to the value of c are well-known. For small positive values of c, the set $C(w_1, w_2, c)$ consists of two disjoint Jordan curves which look like small circles around w_1 and w_2. Increasing c, these two components become more and more egg-shaped until they meet each other in the middle of the line segment between w_1 and w_2 for $c = |w_1 - w_2|/2$. The figure-eight-shaped set

$$L(w_1, w_2) := C(w_1, w_2, |w_1 - w_2|/2)$$

is a lemniscate. For larger values $c > |w_1 - w_2|/2$, the Cassini ovals $C(w_1, w_2, c)$ consist of one component which first looks like a sand glass, then like an ellipse and finally like a large circle (cf. [14, p. 71f.]).

In equation (2), the factor $|1 - \overline{\alpha}z|$ on the right-hand side acts as an "error term" so that the sets

$$\widetilde{C}(0, \alpha, r) := \{ z \in \mathbb{D} : |z|\,|z - \alpha| = r\,|1 - \overline{\alpha}z| \}$$

look like "deformed" Cassini ovals. As z_1 is a critical point of B, the deformed lemniscate $\widetilde{L}(0, \alpha)$ is reached for $r = |B(z_1)|$, i.e. we have

$$\widetilde{L}(0, \alpha) := \{ z \in \mathbb{D} : |B(z)| = |B(z_1)| \}.$$

Let W_1 and W_2 be the components of the open set $\{ z \in \mathbb{D} : |B(z)| < |B(z_1)| \}$ which contain 0 and α, respectively. On the left-hand side of Figure 3, a plot of several deformed Cassini ovals is displayed in case of $\alpha = 0.4 + 0.6i$. The right-hand side of Figure 3 shows a schematic plot of the deformed lemniscate $\widetilde{L}(0, \alpha)$ and the sets W_1 and W_2.

LEMMA 3.2 W_1 *is invariant under B and we have $B(W_2) \subset W_1$.*
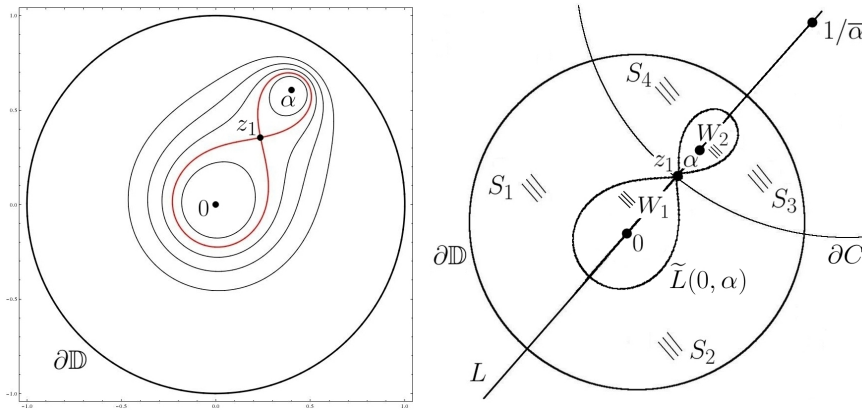
Figure 3.

Proof: For $z \in W_1$, it follows that $|B(B(z))| \leq |B(z)| < |B(z_1)|$. Hence, we obtain $B(z) \in W_1 \cup W_2$ and thus $B(W_1) \subset W_1 \cup W_2$. We have $0 = B(0) \in B(W_1)$, and the continuity of B and the connectedness of W_1 imply that $B(W_1)$ is also connected. According to $B(z) \in B(W_1)$, there exists a path in $B(W_1) \subset W_1 \cup W_2$ which connects 0 and $B(z)$. As this is not possible for $B(z) \in W_2$, we obtain $B(z) \in W_1$. Because of $0 = B(\alpha) \in B(W_2)$, the inclusion $B(W_2) \subset W_1$ can be proved in exactly the same way. $\qquad\square$

It is well-known that we have $B(\mathbb{D}) = \mathbb{D}$ as well as $B(\mathbb{C}_\infty \backslash \overline{\mathbb{D}}) = \mathbb{C}_\infty \backslash \overline{\mathbb{D}}$ and that the origin and the point at infinity are attracting fixed points of B (cf. [13, Problem 7-b on p. 70]). For this reason, the classification theorem of Fatou components implies locally uniform convergence $B^n \to 0$ on $\mathbb{D}$ (see e.g. [2, p. 163]).

THEOREM 3.3   *Comeagre many functions in $M_\infty(\mathbb{C}^*)$ are $M_\infty(W_1^*)$-universal and $M_\infty(W_2 \backslash \{\alpha\})$-universal for C_B.*

Proof: We denote by $\mathcal{G}_1$ the set of all functions in $M_\infty(\mathbb{C}^*)$ which are $M_\infty(W_1^*)$-universal for C_B and by $\mathcal{G}_2$ the set of all functions in $M_\infty(\mathbb{C}^*)$ which are $M_\infty(W_2 \backslash \{\alpha\})$-universal for C_B.

  i) Lemma 3.1 and Lemma 3.2 imply that $B|_{W_1^*} : W_1^* \to W_1^*$ is injective. As we have locally uniform convergence $B^n|_{W_1^*} \to 0$ on $\mathbb{D}$, Theorem 2.3 yields that $\mathcal{G}_1$ is comeagre in $M_\infty(\mathbb{C}^*)$.

  ii) Firstly, we show by induction that B^n is injective on W_2 for all $n \in \mathbb{N}$. For $n = 1$, this follows from Lemma 3.1. Now, let B^n be injective on W_2 and let $z, w \in W_2$ with $B^{n+1}(z) = B^{n+1}(w)$, i.e. $B(B^n(z)) = B(B^n(w))$. As we have $B^n(z), B^n(w) \in W_1$ by Lemma 3.2 and as B is injective on W_1 due to Lemma 3.1, we obtain $B^n(z) = B^n(w)$ and hence $z = w$. Considering the open sets $D := W_1^* \cup (W_2 \backslash \{\alpha\})$ and $U := W_2 \backslash \{\alpha\} \subset D$, Lemma 3.2 yields $B(D) \subset W_1^* \subset D$. According to the injectivity of all iterates B^n on U and the locally uniform convergence $B^n|_U \to 0$, Theorem 2.3 yields that $\mathcal{G}_2$ is comeagre in $M_\infty(\mathbb{C}^*)$. $\qquad\square$

As indicated at the beginning of this section, the statement of Theorem 3.3 can be generalised if the symbol of the composition operator is a rational function ϕ of degree $d_\phi \geq 2$ which has an attracting fixed point at the origin. In order to formulate this result, we consider the case that we have $d_\phi = 2$ and that $0 \in G \subset \mathbb{C}$, $G \neq \mathbb{C}$,

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is a simply connected component of F_ϕ that is completely invariant under ϕ. In this situation, one can show that there exists some $\alpha \in \mathbb{D}^*$ such that $\phi|_G$ is conjugated to $B_\alpha|_{\mathbb{D}}$ (cf. [11, Corollary 5.2.2] and see [11, Example 5.2.3] for diverse examples of such situations). Denoting by z_α the critical point of B_α in $\mathbb{D}$ and considering the open sets $V_{1,\alpha} := \psi^{-1}(W_{1,\alpha})$ as well as $V_{2,\alpha} := \psi^{-1}(W_{2,\alpha})$, where $W_{1,\alpha}$ and $W_{2,\alpha}$ are the components of $\{z \in \mathbb{D} : |B_\alpha(z)| < |B_\alpha(z_\alpha)|\}$ which contain 0 and α, respectively, Theorem 3.3 implies the following statement:

THEOREM 3.4   *In the above situation, comeagre many functions in $M_\infty(\mathbb{C}^*)$ are $M_\infty(V_{1,\alpha}^*)$-universal and $M_\infty(V_{2,\alpha} \backslash \{\psi^{-1}(\alpha)\})$-universal for C_ϕ.*

Acknowledgement

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