The harmonic Faber operator and approximate solutions of Dirichlet problems

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September 29, 2021

Abstract

We study Faber-Fourier series for harmonic functions. It is shown that, in the case of Jordan domains with piecewise Dini-smooth boundary without cusps, the corresponding series of harmonic polynomials converge uniformly for Hölder-continuous functions defined on the boundary of the domain. This results in a constructive approach for the approximate solution of Dirichlet problems by harmonic polynomials in this special situation. Numerical examples for ellipses and squares are given.

Keywords: Faber series, harmonic polynomials, Dirichlet problem
2010 Mathematics Subject Classification: 30E10, 41A58

1 Introduction

For a non-empty compact set K in the complex plane let $C(K)$ denote the space of continuous complex-valued functions on K. We write $a(K)$ for the subspace of all $u \in C(K)$ such that $u|_{K^\circ}$, with K° the interior of K, is harmonic, and $A(K)$ for the subspace of all $h \in C(K)$ such that $h|_{K^\circ}$ is holomorphic. Moreover, let D be a bounded domain in the complex plane and let ω denote a harmonic measure of D. If D is regular (see e.g. [15]), then for $f \in C(\partial D)$ the unique solution of the Dirichlet problem $\Delta u = 0$ in D and $u|_{\partial D} = f$ is given by the Poisson integral

$$u(z) = (P_D f)(z) = \int_{\partial D} f(\zeta)\, d\omega(z,\zeta) \quad (z \in D)$$

and $P_D : C(\partial D) \to a(\overline{D})$ turns out to be an isometric isomorphism (see again e.g. [15]). In the case of the unit disc $D = \mathbb{D}$ we have

$$d\omega(z,\zeta) = \frac{1-|z|^2}{|\zeta - z|^2}\, dm(\zeta) = \mathrm{Re}\Big(\frac{\zeta + z}{\zeta - z}\Big)\, dm(\zeta),$$

where m denotes the normalised arc length measure on the unit circle $\mathbb{T}$. By expanding the Poisson kernel in a geometric series and writing $e_k(z) := z^k$ for

$k \in \mathbb{N}_0$ and $e_{-k}(z) := \overline{z}^k$ for $k \in \mathbb{N}$, one obtains

$$u = Pf := P_{\mathbb{D}} f = \sum_{\nu=-\infty}^{\infty} \widehat{f}(\nu) e_\nu$$

with $\widehat{f}(k)$ the k-th Fourier coefficient of the boundary function f. So we have a series expansion in the harmonic monomials e_k that converges locally uniformly in $\mathbb{D}$. If the Fourier series of f converges uniformly on $\mathbb{T}$, the maximum principle implies that the harmonic polynomials $\sum_{\nu=-n}^{n} \widehat{f}(\nu) e_\nu$ converge uniformly on $\overline{\mathbb{D}}$ to the solution of the Dirichlet problem. According to the Dini-Lipschitz theorem, this holds in particular if f is Hölder continuous. Our aim is to find similar series solutions of Dirichlet problems for more general domains $D \subset \mathbb{C}$. A well-known approach for holomorphic functions is the expansion in a Faber series (see e.g. [2], [4], [16]). We consider, more generally, harmonic Faber series, that is, series expansions in Faber polynomials F_n and conjugate (harmonic) Faber polynomials $\overline{F_n}$ (cf. [1], [16, pp. 280]), where we make systematic use of the extended (harmonic) Faber operator (see [12]).

2 Harmonic Faber series

Let $K \subset \mathbb{C}$ be a compact continuum with a connected complement $\mathbb{C}_\infty \backslash K$. According to the Riemann mapping theorem, there is a unique conformal mapping $\psi := \psi_K \colon \mathbb{C}_\infty \backslash \overline{\mathbb{D}} \to \mathbb{C}_\infty \backslash K$ with

$$\psi(w) = c \cdot w + c_0 + \sum_{\nu=1}^{\infty} c_{-\nu} w^{-\nu} \qquad \left(w \in \mathbb{C}_\infty \backslash \overline{\mathbb{D}}\right)$$

and $c = c_K > 0$. For the inverse function φ of ψ one has

$$\varphi(\xi) = d \cdot \xi + d_0 + \sum_{\nu=1}^{\infty} d_{-\nu} \xi^{-\nu} \qquad (|\xi| > \sup\{|w| : w \in K\})$$

where $d = 1/c$ and, more generally,

$$\varphi^n(\xi) = d^n \cdot \xi^n + \sum_{\nu=0}^{n-1} d_{\nu,n} \xi^\nu + \sum_{\nu=1}^{\infty} d_{-\nu,n} \xi^{-\nu} \qquad (|\xi| > \sup\{|w| : w \in K\}).$$

For $n \in \mathbb{N}$, the n-th Faber polynomial with respect to K is defined by

$$F_n(z) := F_{n,K}(z) := d^n \cdot z^n + \sum_{\nu=0}^{n-1} d_{\nu,n} z^\nu \qquad (z \in \mathbb{C}). \tag{1}$$

It is well-known that, with $F_0 = 1$,

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{\nu=0}^{\infty} \frac{F_{\nu,K}(z)}{w^{\nu+1}} \qquad (z \in K,\ |w| > 1). \tag{2}$$

We put $F_{-n,K} := \overline{F_{n,K}}$ for $n \in \mathbb{N}$.

In the sequel, we restrict our consideration to compact sets K that are the closure of a Jordan domain D. If we fix $a \in D$, then there is a unique Riemann mapping $\varphi_0 : D \to \mathbb{D}$ with $\varphi_0(a) = 0$ and $\varphi'(a) > 0$. Moreover, φ_0 extends to a homeomorphism from K to $\overline{\mathbb{D}}$, which we also denote by φ_0. It is easily seen that here, with ψ_0 the inverse of φ_0,

$$P_D f = P(f \circ (\psi_0|_{\mathbb{T}})) \circ \varphi_0.$$

Let $\widehat{g}(k)$ denote the k-th Fourier coefficient of $g \in C(\mathbb{T})$, for $k \in \mathbb{Z}$. If $\Gamma = \partial K$ is piecewise Dini-smooth, then Γ is of bounded secant variation (see [5, Theorem 4]) and the Faber operator T_D, defined for harmonic polynomials p by

$$p = \sum_{\nu=-N}^{N} \widehat{p}(\nu) e_\nu \mapsto \sum_{\nu=-N}^{N} \widehat{p}(\nu) F_\nu,$$

extends to a continuous linear operator $T = T_D : C(\mathbb{T}) \to a(K)$ (see [12, Theorem 1]). Moreover, if $g \in C(\mathbb{T})$ has a uniformly convergent Fourier series then

$$T_D g = \sum_{\nu=-\infty}^{\infty} \widehat{g}(\nu) F_\nu = \lim_{n\to\infty} \sum_{\nu=-n}^{n} \widehat{g}(\nu) F_\nu \tag{3}$$

with uniform convergence on K and

$$\max_K \Big| T_D g - \sum_{\nu=-n}^{n} \widehat{g}(\nu) F_\nu \Big| \leq \|T_D\| \cdot \max_{\mathbb{T}} \Big| g - \sum_{\nu=-n}^{n} \widehat{g}(\nu) e_\nu \Big|.$$

Note that $\widehat{g}(k) = 0$ for $k \leq 0$ if $g \in A(\overline{\mathbb{D}})$. Identifying, as usual, the functions in $A(\overline{\mathbb{D}})$ with their boundary functions defined on $\mathbb{T}$, we recall that the classical Faber operator $T_D|_{A(\overline{\mathbb{D}})}$ is injective and that $u \in T_D(A(\overline{\mathbb{D}}))$ if and only if the Cauchy integral $c = c(u \circ (\psi|_{\mathbb{T}}))$ of $u \circ (\psi|_{\mathbb{T}})$ belongs to $A(\overline{\mathbb{D}})$ (see e.g. [4]). In the latter case, we have $u = T_D c$. According to Privalov's theorem, if $u \circ (\psi|_{\mathbb{T}})$ is Hölder continuous on $\mathbb{T}$, its Cauchy integral is Hölder continuous on $\overline{\mathbb{D}}$ and thus belongs to $A(\overline{\mathbb{D}})$.

Let $h(D)$ and $H(D)$, respectively, denote the spaces of harmonic functions in D and holomorphic functions in D. It is easily seen that $h(D) = H(D) \oplus \overline{H}(D)$, where

$$\overline{H}(D) := \{\overline{u} : u(a) = 0,\ u \in H(D)\}$$

is the space of anti-holomorphic functions in D vanishing at a. For $f \in C(\Gamma)$ we write

$$P_D f = Q_D f + R_D f,$$

where $Q_D f \in H(D)$ and $R_D f \in \overline{H}(D)$. Moreover, for $\alpha \in (0, 1]$ let $a_\alpha(K)$ denote the space of functions in $a(K)$ which satisfy a Hölder condition of order α on Γ and, similarly, let $A_\alpha(K)$ denote the space of functions in $A(K)$ which satisfy a Hölder condition of order α on Γ. Equipped with the corresponding

Hölder-norms, $a_\alpha(K)$ and $A_\alpha(K)$ become Banach spaces. In [17] it is shown that functions in $A_\alpha(K)$ satisfy a Hölder condition also on K. We set $a_+(K) := \bigcup_{\alpha>0} a_\alpha(K)$ and $A_+(K) := \bigcup_{\alpha>0} A_\alpha(K)$.

If Γ has no cusps, then it satisfies an inner and an outer wedge condition at each corner (and thus at each point). According to results of Lesley (see [10] and [11]), the conformal mappings ψ_0 and φ_0 are Hölder continuous. With

$$\overline{A_+}(K) := \{\overline{u} : u(a) = 0,\, u \in A_+(K)\}$$

and according to continuity properties of Cauchy integrals (Privalov's theorem) this implies (cf. [13, Prop. 2.28])

$$a_+(K) = A_+(K) \oplus \overline{A_+}(K). \tag{4}$$

We write $C_+(\Gamma)$ for the space of functions in $C(\Gamma)$ which are Hölder continuous of some order $\alpha > 0$. Then (4) implies the injectivity of $T_D|_{C_+(\mathbb{T})}$ (cf. [13, Proposition 2.30]).

Let now $f \in C_+(\Gamma)$. Then f is Hölder continuous of some order $\alpha > 0$ and thus $P_D f \in a_\alpha(K)$. From (4) we have $Q_D f \in A_+(K)$ and $R_D f \in \overline{A_+}(K)$. Hence, with $Q := Q_{\mathbb{D}}$, and $R = R_{\mathbb{D}}$ the functions $Q(Q_D f \circ (\psi|_{\mathbb{T}}))$ and $R(R_D f \circ (\psi|_{\mathbb{T}}))$ belong to $C_+(\mathbb{T})$, so that for

$$S_D f := Q(Q_D f \circ (\psi|_{\mathbb{T}})) + R(R_D f \circ (\psi|_{\mathbb{T}}))$$

we have $S_D(C_+(\Gamma)) \subset C_+(\mathbb{T})$. From [12, Theorem 3] we obtain $(T_D S_D f)|_\Gamma = f$ and hence $T_D S_D f = P_D f$, by uniqueness of the solution of the Dirichlet problem. The Dini-Lipschitz theorem implies that the Fourier series of $S_D f$ converges uniformly on $\mathbb{T}$. Summing up, we obtain

Theorem 2.1. *Let D be a Jordan domain with piecewise Dini-smooth boundary having no cusps. If $f \in C_+(\Gamma)$ then*

$$P_D f = T_D S_D f = \sum_{\nu=-\infty}^{\infty} (S_D f)\hat{}(\nu) F_\nu \tag{5}$$

with uniform convergence on K.

The theorem shows that, for domains D with piecewise Dini-smooth boundary having no cusps, the Dirchlet problem with boundary function $f \in C_+(\Gamma)$ can be solved approximately by partial sums of (5), this means, by harmonic polynomials matching the boundary function up to a prescribed (absolute) error in the uniform norm, where the coefficients are given as Fourier coefficients of $S_D f$. Similar approaches to solve Dirichlet problems are described in [1] and [16, pp. 280], where the boundary functions are less restricted but Γ is required to be analytic or of sufficient smoothness. Note that Theorem 2.1 applies in particular to the case of a polygonal domain.

In view of Theorem 2.1 immediately the question arises how the Fourier transform $(S_D f)\hat{}$ can be calculated in terms of f and D without harmonic conjugates involved.

We have

$$P_D f = P(f \circ \psi_0|_{\mathbb{T}}) \circ \varphi_0 = \int_{\mathbb{T}} f(\psi_0(\zeta))\, \mathrm{Re}\Big(\frac{\zeta + \varphi_0}{\zeta - \varphi_0}\Big)\, dm(\zeta)$$

on D. Since

$$\mathrm{Re}\Big(\frac{\zeta + \varphi_0}{\zeta - \varphi_0}\Big) = 1 + \sum_{\mu=1}^{\infty} \varphi_0^\mu / \zeta^\mu + \sum_{\mu=1}^{\infty} \overline{\varphi_0}^\mu / \overline{\zeta}^\mu$$

with locally uniform convergence on D we obtain

$$T_D S_D f = P_D f = (f \circ \psi_0)\widehat{\ }(0) + \sum_{\mu=1}^{\infty} (f \circ \psi_0)\widehat{\ }(\mu) \varphi_0^\mu + \sum_{\mu=1}^{\infty} (f \circ \psi_0)\widehat{\ }(-\mu) \overline{\varphi_0}^\mu .$$

Expansion of $\varphi_0^\mu \in A_+(K)$ into a Faber series leads to

$$\varphi_0^\mu(z) = \sum_{\nu=0}^{\infty} (\varphi_0^\mu \circ \psi)\widehat{\ }(\nu) F_\nu(z)$$

with uniform convergence on K. Since $\varphi_0 \circ \psi$ is bounded by 1 on Γ, the same holds for the Fourier coefficients $(\varphi_0^\mu \circ \psi)\widehat{\ }(\nu)$. If $(F_\nu(z))$ is absolutely summable for some $z \in K$ (which is e.g. the case if ψ'' beongs to the Hardy space H^1; see [16, p. 83]), and if $((f \circ \psi_0)\widehat{\ }(\mu))$ is absolutely summable (which is the case if $f \circ \psi_0 \in \mathrm{Lip}(\alpha)$ for some $\alpha > 1/2$ by Bernstein's Theorem, and if $f \circ \psi_0$ is in $C_+(\mathbb{T})$ and of bounded variation by a result of Zygmund; see e.g [9]), then by interchanging the order of summation and comparing the coefficients (which is allowed due to the injectivity of $T_D|_{C_+(\mathbb{T})}$) we obtain

$$(S_D f)\widehat{\ }(k) = \sum_{\mu=-\infty}^{\infty} (f \circ \psi_0)\widehat{\ }(\mu) \cdot (\varphi_0^\mu \circ \psi)\widehat{\ }(k)$$

with absolute convergence. The important feature is that the required information concerning the boundary function f is reduced to the Fourier coefficients $(f \circ \psi_0)\widehat{\ }(\mu)$. In particular, no harmonic conjugates are needed here. The Fourier transforms $(\varphi_0^\mu \circ \psi)\widehat{\ }$ depend only on the domain D, but not on f. So, once computed, variations in the boundary function f only require the evaluation of $(f \circ \psi_0)\widehat{\ }(\mu)$. Since we are restricted to Fourier coefficients, Fast Fourier Transform (FFT) turns out to be a quite efficient tool.

An further approach (cf. [16, pp. 280], [13]) to calculate $(S_D f)\widehat{\ }$ in terms of $f \circ \psi_0$ is based on the expansion of the Schwarz-kernel

$$s(\zeta, z) := \frac{\zeta + \varphi_0(z)}{\zeta - \varphi_0(z)}$$

into a Faber series. If $|w| > 1$ then $s(w, \cdot) \in A_+(\overline{\mathbb{D}})$ and the Faber series

$$s(w, z) = \sum_{\mu=0}^{\infty} a_\mu(w) F_\mu(z),$$

with

$$a_k(w) := (s(w,\cdot)\circ\psi)\widehat{\ }(k) \qquad (k\in\mathbb{N}_0)$$

converges uniformly on K. Moreover, the parameter integrals a_k are holomorphic in $\mathbb{C}\backslash\overline{\mathbb{D}}$. If the a_k have radial boundary values $a_k(\zeta)$ at almost all $\zeta\in\mathbb{T}$ and if for some $z\in D$ we have $s(\zeta,z)=\sum_{\mu=0}^{\infty} a_\mu(\zeta)F_\mu(z)$ with "reasonable" convergence with respect to ζ, then the Fourier transform $(S_Df)\widehat{\ }$ may be deduced from

$$(T_DS_Df)(z) = \int (f\circ\psi_0)\,\mathrm{Re}(s(\cdot,z))\,dm = \sum_{\mu=-\infty}^{\infty} F_\mu(z)\int (f\circ\psi_0)\,b_\mu\,dm,$$

where $b_0 := \mathrm{Re}(a_0)$, $b_k := a_k/2$ for $k>0$ and $b_k := \overline{b_{-k}}$ for $k<0$, namely,

$$(S_Df)\widehat{\ }(k) = \int (f\circ\psi_0)b_k\,dm \quad (k\in\mathbb{Z}). \tag{6}$$

Again, it is seen that the dependence on f is only in form of $f\circ\psi_0$ and in particular no harmonic conjugate is involved. Also, once the a_k (and then also the b_k) are evaluated, for varying f the computation of $(S_Df)\widehat{\ }(k)$ can be done efficiently by numerical integration.

If Γ is piecewise analytic, that is, if ψ extends holomorphically beyond $\mathbb{T}$ except for finitely many points $\zeta_1,\ldots,\zeta_m$, then, due to deformation of the contour of integration $\mathbb{T}$ underlying the Fourier coefficients $a_k(w)$, the functions a_k also extend holomorphically beyond $\mathbb{T}$ except for the points $\varphi(\psi_0(\zeta_1)),\ldots,\varphi(\psi_0(\zeta_m))$.

If Γ is analytic, then $s(\zeta,z)=\sum_{\mu=0}^{\infty} a_\mu(\zeta)F_\mu(z)$ holds uniformly with respect to ζ, for each $z\in D$ (see [13]). Since the a_k are given as Fourier coefficients in terms of the the Schwarz kernel, again FFT can be employed for efficient computation.

3 Example I: Ellipse

For fixed $R>1$ let $D=\left\{z\in\mathbb{C}\colon (\mathrm{Re}(z)/a)^2+(\mathrm{Im}(z)/b)^2<1\right\}$ be the domain bounded by the ellipse with semi-axes

$$a=\frac{1}{2}\left(R+\frac{1}{R}\right),\quad b=\frac{1}{2}\left(R-\frac{1}{R}\right).$$

Then $\psi\colon \mathbb{C}_\infty\backslash\overline{\mathbb{D}}\to\mathbb{C}_\infty\backslash K$ is given by

$$\psi(w) := \frac{1}{2}\left(Rw+\frac{1}{Rw}\right) \qquad \left(w\in\mathbb{C}_\infty\backslash\overline{\mathbb{D}}\right)$$

and

$$F_{n,K}=\frac{2}{R^n}T_n \qquad (n\in\mathbb{N})$$

where T_n denotes the Chebyshev polynomial of degree n. Since the Chebyshev polynomials can be computed efficiently, the main task for evaluating the n-partial sum of (5) is in the approximative computation of the Fourier coefficients

$$c_k(f) := (S_D f)\widehat{\ }(k)$$

for $|k| \leq n$. Since the domain is analytically bounded, we can compute the $c_k(f)$ with the aid of Equation (6). So we need to have knowledge about b_k and hence about the conformal mappings $\varphi_0 : D \to \mathbb{D}$ and $\psi_0 : \mathbb{D} \to D$, given here in terms of elliptic integrals. The incomplete elliptic integral of the first kind F is defined as

$$F\left(z, t^2\right) = \int_0^z \frac{dx}{\sqrt{\left(1-x^2\right)\left(1-t^2x^2\right)}}. \tag{7}$$

The square root is to be regarded as $\sqrt{1-x}\sqrt{1+x}\sqrt{1-tx}\sqrt{1+tx}$; the arguments are determined such that each factor is equal to 1 for $x = 0$. We set $K\left(t^2\right) := F\left(1, t^2\right)$ for $t^2 \neq 1$; then K is called the complete elliptic integral of the first kind. The inverse of $F\left(\cdot\,, t^2\right)$ is given by $\operatorname{sn}\left(\cdot\,, t^2\right)$ for $t^2 \neq 1$, with sn denoting the sinus amplitudinis.

As one can find in [8, p. 322] and [14, p. 296], φ_0 has the representation

$$\varphi_0\left(z\right) = \sqrt{s}\cdot \operatorname{sn}\Big(\frac{2K\left(s^2\right)}{\pi}\arcsin\left(z\right); s^2\Big) \qquad \left(z \in D\right). \tag{8}$$

Here, $\arcsin\left(z\right) = -i\log\left(iz + \sqrt{1-z^2}\right)$ where $\sqrt{1-z^2}$ has to be understood as product $\sqrt{1-z}\sqrt{1+z}$ and the branches of the square roots are taken such that each factor is equal to 1 for $z = 0$, and the principal value of the logarithm is taken. The modulus $s \in (0,1)$ can be computed via the equation

$$\frac{\pi K\left(1-s^2\right)}{2K\left(s^2\right)} = 2\operatorname{arsinh}\left(b\right).$$

Equation (8) implies

$$\psi_0\left(w\right) = \sin\left(\frac{\pi}{2K\left(s^2\right)}F\left(\frac{w}{\sqrt{s}}, s^2\right)\right) \qquad \left(w \in \mathbb{D}\right).$$

For numerical purposes, FFT provides an efficient and stable approach for the evaluation of the Fourier coefficients $a_k(\zeta)$ for $k \in \mathbb{N}_0$ and $\zeta \in \mathbb{T}$ (cf. [6]).

As for example, we approximately solve several Dirichlet problems with varying the boundary function f where we fix the ellipse with semi axis $a = 5/4$ and $b = 3/4$. Figure 1 and Figure 2, respectively, show the 10-th partial sum of the Faber expansion (5) for $f(z) = \left|\operatorname{Re}\left(z\right)\right|^{3/2}$ and for $f\left(z\right) = \left|\operatorname{Re}\left(z\right)\right|$ with the exact boundary function f in red. The non-smoothness at $\pm i3/4$ in the second case naturally causes a larger error near these points. Furthermore, we consider a boundary functions which has isolated singularities inside or outside the ellipse if considered as (rational) function in $\mathbb{C}$, namely $f\left(z\right) = \operatorname{Re}(1/(1-z^4))$ having the singularities ± 1 (in the interior of the ellipse) and $\pm i$ (in the exterior of the ellipse). In this case, the partial sum is of degree 20. The result can be seen in Figure 3.

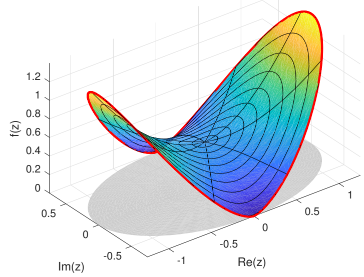

Figure 1: Plot of f in the case $f(z) = |\mathrm{Re}(z)|^{3/2}$

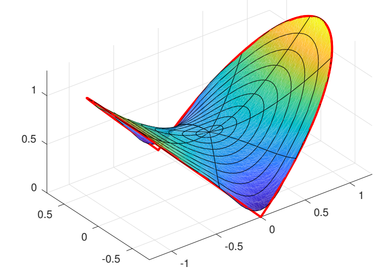

Figure 2: Approximation of u in the case $f(z) = |\mathrm{Re}(z)|$

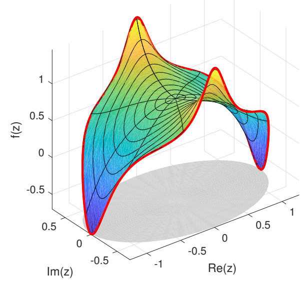

Figure 3: Approximation of u in the case $f(z) = \mathrm{Re}\left(1/(1-z^4)\right)$

4 Example II: Square

We consider the interior D of the square with corners in ± 1 and $\pm i$. Although D is not bounded by an analytic Jordan curve, we apply the above method to compute an approximate solution of a given Dirichlet problem. To do so, we have to know about the conformal mappings φ_0, ψ_0 and ψ. Since D is a square, these functions are given by Schwarz-Christoffel mappings (see e.g. [7, p. 411ff.]): We have

$$\psi_0(w) = C\int_0^w \frac{dz}{\sqrt{1-z^4}} = CF(w,-1) \qquad (w \in \mathbb{D})$$

where C is determined by

$$1 = C\int_0^1 \frac{dw}{\sqrt{1-w^4}}.$$

That leads us to

$$C = \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{2}\right)}$$

with Γ denoting the gamma function. For the inverse function φ_0, we obtain the representation

$$\varphi_0(z) = \operatorname{sn}(z/c,-1) \qquad (z \in D).$$

Further, the function ϕ defined by

$$\phi(w) = C_1\int_{w_0}^w \frac{\sqrt{1-z^4}}{z^2}dz + C_2 \qquad (w \in \mathbb{D}),$$

where C_1, C_2 and $w_0 \neq 0$ are chosen so that $\phi(\pm 1) = \pm 1$ and $\phi(\pm i) = \pm i$, maps $\mathbb{D}$ conformally onto $\mathbb{C}_\infty \backslash K$ with $\phi(0) = \infty$. One computes

$$\int_{w_0}^w \frac{\sqrt{1-z^4}}{z^2}dz = -\left(\frac{\sqrt{1-z^4}}{z} - 2\left(E(z,-1) - F(z,-1)\right)\right)\Bigg|_{w_0}^{w}$$

where

$$E\left(z,t^2\right) = \int_0^z \sqrt{\frac{(1-t^2x^2)}{(1-x^2)}}dx$$

denotes the incomplete elliptic integral of the second kind. Here, the root means

$$\frac{\sqrt{(1-tx)}\sqrt{(1+tx)}}{\sqrt{1-x}\sqrt{1+x}}$$

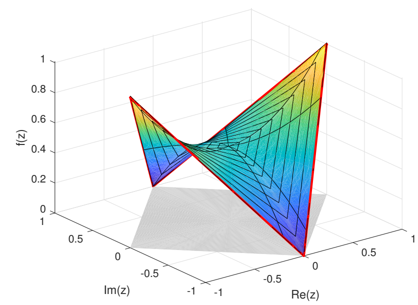

Figure 4: Approximation of u in the case $f(z) = |\mathrm{Re}(z)|$

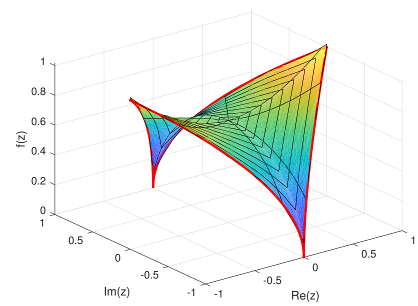

Figure 5: Approximation of u in the case $f(z) = \sqrt{|\mathrm{Re}(z)|}$

where the branches of the roots are taken such that each factor is equal to 1 for $x = 0$. By $\psi(w) = \phi(1/w)$, we obtain the looked-for ψ.

As in the case of analytically bounded domains, the a_k do not depend on the boundary function f. However, now we are faced with the problem of having corners in Γ. But still Γ is piecewise analytic, so except for the preimages of the corners under $\varphi \circ \psi_0$, the functions a_k exist on $\mathbb{T}$. By modifying the contour of integration appropriately, we can approximately evaluate $a_k(\zeta)$ by numerical integration (for details and corresponding MATLAB codes see [13]). Then the Fourier coefficients $c_k(f) := (S_D f)\widehat{\ }(k)$ can be achieved from (6) by numerical integration.

Computation of the partial sums of (5) also requires evaluation of the Faber polynomials $F_{n,K}$. We have calculated the $F_{n,K}$ with the Schwarz-Christoffel toolbox for MATLAB, which is established by Driscoll and introduced in [3]. Figures 4 to 6 show examples of the 10-th partial sums for different boundary functions f as well as the exact functions in red.

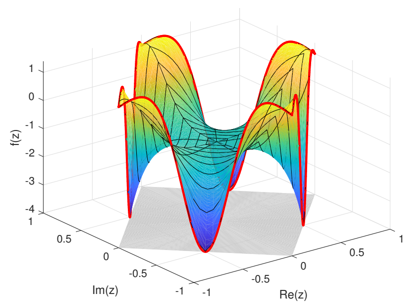

Figure 6: Approximation of u in the case $f(z) = \text{Re}\left(1/z^4\right)$

Data Availability Statement The authors confirm that the data supporting the findings of this work are available within the article.

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