Generic boundary behaviour of Taylor series in Banach spaces of holomorphic functions

Jürgen Müller

University of Trier, FB IV, Mathematics, 54286 Trier, Germany
jmueller@uni-trier.de

Abstract: For several classical Banach spaces X of functions holomorphic in the unit disc we consider the question to which extent, for some $f \in X$, the sequence $(S_n f)$ of the partial sums of the Taylor series about 0 behaves erratically on compact sets outside of the unit disc. The problem is closely related to a question about simultaneous polynomial approximation.

1 Universality of Taylor sections

Let $\mathbb{C}_\infty$ be the extended complex plane. For an open set $\Omega \subset \mathbb{C}_\infty$ we denote by $H(\Omega)$ the Fréchet space of functions holomorphic in Ω (and vanishing at ∞ if $\infty \in \Omega$) endowed with the topology of locally uniform convergence. If $0 \in \Omega$ and $f \in H(\Omega)$, we write with $P_k(z) := z^k$

$$S_n f := \sum_{\nu=0}^{n} a_\nu(f) P_k$$

for the n-th partial sum of the Taylor expansion of f about 0. A classical question in complex analysis is how the partial sums $S_n f$ behave outside the disc of convergence and in particular on the boundary of the disc. Based on Baire's category theorem, it can be shown that for functions f holomorphic in the unit disc $\mathbb{D}$ generically the sequence $(S_n f)$ turns out to be "maximally divergent" outside of $\mathbb{D}$. In order to clarify in which sense maximal divergence is understood, we introduce some notations.

For E compact in $\mathbb{C}$ we write

$$A(E) := \{h \in C(E) : h \text{ holomorphic in } E^\circ\}.$$

If E has connected complement then, according to Mergelyan's theorem, $A(E)$ is the closure of the set of polynomials in $C(E)$, where $C(E)$ is endowed with the uniform norm $||\cdot||_E$. For Λ an infinite subset of $\mathbb{N}$ and $f \in H(\mathbb{D})$, we say that the sequence $(S_n f)$ is Λ-**universal** on E if for all $h \in A(E)$ a subsequence of $(S_n f)_{n\in\Lambda}$ tends to h uniformly on E, in other words, if $\{S_n f : n \in \Lambda\}$ is dense in $A(E)$. In 1996, Nestoridis proved that for each Λ a residual set of functions $f \in H(\mathbb{D})$ have the property that $(S_n f)$ is Λ-universal on each compact set $E \subset \mathbb{C} \setminus \mathbb{D}$ with connected complement (see [**22**]). For corresponding results we refer in particular to the expository articles [**10**] and [**16**]. For results on universal series in a more general framework see also [**2**].

If X is a Fréchet space of functions in $H(\mathbb{D})$ which is continuously embedded in $H(\mathbb{D})$ we say that a compact set $E \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ is a **set of universality** for X if for all infinite sets $\Lambda \subset \mathbb{N}$ there is a residual set of functions $f \in X$ with the property that the sequence $(S_n f)$ is Λ-universal on E. Moreover, we call E a **set of σ-universality** for X, if an increasing sequence of sets of universality E_k exists with $E = \bigcup_{k\in\mathbb{N}} E_k$. In this

Research Perspectives CRM Barcelona, Fall 2019, vol. 12, in Trends in Mathematics
Springer-Birkhäuser, Basel

case, Baire's theorem implies that for each Λ there is a residual set of $f \in X$ having the property that each continuous function on E is the pointwise limit of some subsequence of $(S_nf)_{n\in\Lambda}$. Nestoridis' result implies that the unit circle $\mathbb{T}$ is a set of σ-universality for $H(\mathbb{D})$.

The situation changes if we consider classical Banach spaces of holomorphic functions in $\mathbb{D}$. Our aim is to study the generic limit behaviour of the Taylor sections S_nf outside of $\mathbb{D}$ and in particular on $\mathbb{T}$ for functions in Hardy spaces H^p, in Bergman spaces A^p and in the Dirichlet space D. We recall that, with m denoting the normalised arc length measure on $\mathbb{T}$ and m_2 the normalised area measure on $\mathbb{D}$ and for $1 \le p < \infty$,

$$\begin{aligned} H^p &= \{f \in H(\mathbb{D}) : \|f\|^p := \sup_{r<1} \int_{\mathbb{T}} |f(r\,\cdot)|^p \, dm < \infty\}, \\ A^p &= \{f \in H(\mathbb{D}) : \|f\|^p := \int_{\mathbb{D}} |f|^p \, dm_2 < \infty\}, \\ D^p &= \{f \in H(\mathbb{D}) : \|f\|^p := \int_{\mathbb{D}} |(\mathrm{id}\, f)'|^p \, dm_2 < \infty\}. \end{aligned}$$

It is known that $D^p \subset H^p \subset A^{2p}$ for all p. According to Fatou's theorem, each $f \in H^p$ has nontangential limits $f^*(\zeta)$ at m-almost all ζ in $\mathbb{T}$ and $f^* \in L^p(\mathbb{T})$. Moreover, the mapping $f \mapsto f^*$ establishes an isometry between H^p and the closure of the polynomials in $L^p(\mathbb{T})$. According to the famous Carleson-Hunt theorem, for each $p > 1$ and each $f \in H^p$ the partial sums S_nf converge to f^* almost everywhere on $\mathbb{T}$. Moreover, due to results of Kolmogorov, for functions in H^1 convergence in measure still holds and hence each subsequence of (S_nf) has a sub-subsequence that converges almost everywhere to the boundary function f^*. In the case of the classical Dirichlet space $D := D^2$, more can be said. We recall that a set $E \subset \mathbb{C}$ is called polar if it has vanishing logarithmic capacity, and that a property is said the be satisfied quasi everywhere if it is satisfied up to a polar set. If f belongs to D then existence of f^* is guaranteed quasi everywhere (Beurling's theorem; see e.g. [**8**, Theorem 3.2.1]) and convergence of (S_nf) to f^* holds in all points $\zeta \in \mathbb{T}$ where $f^*(\zeta)$ exists (see e.g. [**20**, p. 12]).

In contrast, generically in A^p functions do not have non-tangential limits almost everywhere. However, from a result of Shkarin ([**23**]; cf. also [**9**, Corollary 2]) it follows that for all $f \in A^1$ and for all arcs $E \subset \mathbb{T}$ the sequence (S_nf) has at most one limit function $h \in C(E)$ on E. Concerning possible sets of universality, we conclude from the above results:

- No closed $E \subset \mathbb{T}$ with $m(E) > 0$ is a set of universality for any H^p.
- No closed $E \subset \mathbb{T}$ of positive capacity is a set of universality for D.
- No arc $E \subset \mathbb{T}$ is a set of σ-universality for any A^p.

In the converse direction we have (see [**3**], [**21**])

Theorem 1 *Closed sets $E \subset \mathbb{T}$ with $m(E) = 0$ are sets of universality for all H^p and closed polar sets $E \subset \mathbb{T}$ are sets of universality for D. Moreover, there are sets of full measure $E \subset \mathbb{T}$ which are sets of σ-universality for all A^p.*

The proof of Theorem 1 is based on results on simultaneous approximation by polynomials. We write $X \oplus Y$ for the (external) direct sum of two Fréchet spaces X and Y and we say that a compact set $E \subset \mathbb{C} \setminus \mathbb{D}$ is a **set of simultaneous approximation** for the Fréchet space $X \subset H(\mathbb{D})$ if the pairs $(P|_{\mathbb{D}}, P|_E)$, where P ranges over the set of polynomials, form a dense set in the sum $X \oplus A(E)$.

Lemma 2 *Let X be a Fréchet space continuously embedded in $H(\mathbb{D})$ and let $E \subset \mathbb{C} \setminus \mathbb{D}$. If E is a set of simultaneous approximation for X then E is a set of universality for X.*

Proof. Since X is continuously embedded in $H(\mathbb{D})$, the mappings $X \ni f \mapsto a_k(f)P_k|_E \in A(E)$ are continuous. For fixed infinite set $\Lambda \subset \mathbb{N}_0$ we consider the family $(S_n)_{n\in\Lambda}$ (more precisely $f \mapsto S_n f|_E$) of continuous linear mappings from X to $A(E)$. The Universality Criterion (see e.g. [**10**, Theorem 1] or [**11**, Theorem 1.57]) implies that it is sufficient – and necessary – to show that for each pair $(f, g) \in X \oplus A(E)$ and each $\varepsilon > 0$ there exist a polynomial P and an integer $n \in \Lambda$ so that $||f - P||_X < \varepsilon$ and $||g - S_n P||_E < \varepsilon$. Since $S_n P = P$ for all polynomials P and all large n (depending on the degree of P), this is satisfied if E is a set of simultaneous approximation for X. □

While the simultaneous approximation condition is sufficient for the universality property to hold, is turns out to be not necessary in general: Obviously, no non-empty set $E \subset \mathbb{T}$ is a set of universality for the disc algebra $A(\overline{\mathbb{D}})$. On the other hand, finite sets $E \subset \mathbb{T}$ are (exactly the) sets of universality for $A(\overline{\mathbb{D}})$ (see [**13**], [**6**, Theorem 3.3] for the sufficiency and [**4**, Corollary 3.5] for the necessity).

2 Simultaneous approximation by polynomials

Let now $X = (X, \|\cdot\|_X)$ be a Banach space of functions continuous on some set $M \subset \mathbb{C}$. Moreover, we suppose that the polynomials are dense in X and that the sequence $(||P_k||_X^{1/k})_k$ is bounded. In this case we briefly speak of a **regular** space X. By X' we denote the (norm) dual of X and by $H(0)$ the linear space of germs of functions holomorphic at 0. Then the Cauchy transform $K_X : X' \to H(0)$ with respect to X is defined by

$$(K_X\phi)(w) = \sum_{\nu=0}^{\infty} \phi(P_\nu)w^\nu$$

for $|w|$ sufficiently small and $\phi \in X'$. Since the polynomials form a dense set in X, the Cauchy transform K_X is injective. Then the range $X^* = R(K_X)$ of K_X is called **Cauchy dual** of X. The following consequence of the Hahn-Banach theorem (see [**14**, Theorem 1.2], [**6**, Lemma 2.7])) is the basis for our subsequent considerations.

Lemma 3 *Let X and Y be regular. Then $X^* \cap Y^* = \{0\}$ if and only if the pairs (P, P), where P ranges over the set of polynomials, form a dense set in the sum $X \oplus Y$.*

Let $E \subset \mathbb{C}$ be compact with connected complement. Since $A(E)$ is a subspace of $C(E)$, according to the Hahn-Banach theorem and the Riesz representation theorem, each $\phi \in A(E)'$ can be represented (not uniquely) by some complex Borel measure μ supported on E . If μ is an arbitrary representing measure of ϕ then $K_{A(E)}(\phi)$ is the germ at 0 of the Cauchy transform $\widehat{\mu} \in H(\mathbb{C}_\infty \setminus (1/E))$ given by

$$\widehat{\mu}(w) = \int_E \frac{d\mu(z)}{1 - zw} \quad (w \in \mathbb{C}_\infty \setminus (1/E)).$$

Similarly, for $1 < p < \infty$ and $\phi \in (H^p)'$ the Cauchy transform is of the form

$$(K_{H^p}\phi)(w) = \int_{\mathbb{T}} \frac{\overline{h}(z)}{1 - zw}\, dm(z) \quad (w \in \mathbb{D}),$$

for some $h \in H^{p^*}$, where p^* denotes the conjugate exponent. Then $K_{H^p}(\phi) \in H^{p^*}$ and, more precisely, the Cauchy dual $(H^p)^*$ equals H^{p^*} in this sense (cf. [**5**] or [**7**]). In the same way, $(D^p)^* = A^{p^*}$ (see [**5**, p. 88]).

By proving uniqueness of the Cauchy transforms involved and applying Lemma 3, one can show

Theorem 4 *Closed sets $E \subset \mathbb{T}$ with $m(E) = 0$ are sets of simultaneous approximation for all H^p and closed polar sets $E \subset \mathbb{T}$ are sets of simultaneous approximation for D.*

The first statement, originally proved by Havin (see [**12**], cf. [**14**]) is a consequence of the F. and M. Riesz theorem. As I learnd during the SAFAIS conference, the second one is due to Khrushchev and Peller ([**15**]). It was reinvented in [**21**] with a proof using Tumarkin's theorem and the fact that polar sets are removable for $A^2(2\mathbb{D} \setminus (1/E))$. As I realized recently, this proof contains a flaw, which fortunately can be corrected by a very nice argument given by Koosis (see [**19**, Lemma 2]) and published already shortly after [**15**].

The first two statements of Theorem 1 follow immediately from Theorem 4 and Lemma 2. The third statement of Theorem 1 is a consequence of Lemma 2 and the following result on simultaneous approximation in A^p, which is based on a deep theorem of Khrushchev on uniqueness of Cauchy transforms (see [**14**, Theorem 4.1]), and the fact that an increasing sequence (E_k) of Cantor sets $E_k \subset \mathbb{T}$ satisfying Carleson's condition with $m(E_k) \to 1$ as $k \to \infty$ exists (see [**3**]). Recall that a closed set $E \subset \mathbb{T}$ satisfies Carleson's (entropy) condition if $\sum_I m(I) \log(1/m(I)) < \infty$, where the sum ranges over all components $I \subset \mathbb{T}$ of $\mathbb{T} \setminus E$. From [**3**, Theorem 2.2] one obtains

Theorem 5 *Each Cantor set $E \subset \mathbb{T}$ of positive measure satisfying Carleson's condition is a set of simultaneous approximation for all A^p with $1 < p < \infty$.*

In the very recent paper [**18**], Khrushchev proved two new results on universality of Taylor series.[1] Both are based on results on simultaneous approximation that can be seen as extensions of Theorem 4: Firstly, Corollary 1 in [**18**] says that closed sets $E \subset \mathbb{T}$ with $m(E) = 0$ are sets of simultaneous approximation for the quotient space $C(\mathbb{T})/B$, where $B := \{f \in C(\mathbb{T}) : \widehat{f}(n) = 0 \text{ for } n \in \mathbb{N}_0\}$. According to the duality results of Fefferman, Sarason and Stein (see e.g. [**5**, Section 3.5]), this can also be seen as a result on simultaneous approximation for the space $VMOA$. Secondly, replacing logarithmic capacity by corresponding p-capacities, sets of vanishing p-capacity are sets of simultaneous approximation for the analytic Besov spaces $\mathbb{P}_+B_p^{1/p}$, where $1 < p < \infty$ (for $p = 2$, the Dirichlet space case is recovered). Hence, sets $E \subset \mathbb{T}$ of vanishing m-measure are sets of universality for $VMOA$ and sets of vanishing p-capacity are sets of universality for $\mathbb{P}_+B_p^{1/p}$. The latter statement is essentially [**18**, Theorem 3.2]).

In all above results, simultaneous approximation is restricted to the case of compact subsets of the unit circle $\mathbb{T}$. We briefly discuss the situation of more general sets $E \subset \mathbb{C} \setminus \mathbb{D}$, which was considered only recently and is not yet fully clarified.

It is known that compact sets $E \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ with connected complement are sets of universality even for the space $A^\infty(\overline{\mathbb{D}})$ of functions in $H(\mathbb{D})$ having the property that each derivative has a continuous extension to $\overline{\mathbb{D}}$ (see e.g. [**16**, Theorem 4.2]). In view of

[1] The paper appeared online after the SAFAIS conference and I became aware of it while writing the extended abstract.

Theorem 1, a reasonable guess is that compact sets $E \subset \mathbb{C} \backslash \mathbb{D}$ with connected complement and touching $\mathbb{T}$ in a set of vanishing m-measure would be sets of universality for H^p. It turns out that this in not true. More precisely, using an idea of Bayart (see [**1**]), in [**6**] the following result was proved:

Theorem 6 *Let $E \subset \mathbb{C} \setminus \mathbb{D}$ be a compact set containing a rectifiable arc $\gamma : [0,1] \to \mathbb{C} \setminus \mathbb{D}$ with $\gamma(0) \neq \gamma(1)$ and $\gamma(0), \gamma(1) \in \mathbb{T}$. Then for all $f \in H^1$ the sequence $(S_n f)$ is not universal on E.*

Theorem 6 shows that even nice compact sets E touching $\mathbb{T}$ in only two points, as e.g. the subarc of $|z-1| = 1$ lying outside the unit disc, cannot be sets of universality for H^1. A result in the converse direction is also given in [**6**]. Recall that, according to the Cantor-Bendixson theorem, each compact set $A \subset \mathbb{C}$ can be decomposed in unique way as union of a perfect set, called the perfect kernel of A, and a countable set. For compact $E \subset \mathbb{C} \setminus \mathbb{D}$ and $\zeta \in E \cap \mathbb{T}$ let $C_E(\zeta)$ denote the component of E that contains ζ and let P_E be the union of all $C_E(\zeta)$ with ζ ranging over the perfect kernel of $E \cap \mathbb{T}$. Then E is said to satisfy the kernel condition if P_E has vanishing area measure and if, in addition, the set of all $\zeta \in E \cap \mathbb{T}$ with the property that $C_E(\zeta)$ has positive area measure has positive distance to the perfect kernel of $E \cap \mathbb{T}$.

Theorem 7 *Let $E \subset \mathbb{C} \setminus \mathbb{D}$ be a compact set with connected complement which satisfies the kernel condition. Moreover, suppose that no component of E touches $\mathbb{T}$ in more than one point. If $m(E \cap \mathbb{T}) = 0$ then E is a set of simultaneous approximation for all H^p and if $E \cap \mathbb{T}$ is polar then E is a set of simultaneous approximation for D.*

The first part of Theorem 7 is proved in [**6**], the second can be proved in essentially the same way, where in the final step the uniqueness argument leading to part 2 of Theorem 4 has to be repeated.

According to the results of Section 1, the conditions on $E \cap \mathbb{T}$ turn out to be necessary and, due to Theorem 6, the condition that no component of E touches $\mathbb{T}$ in more than one point is a natural one. It is not known if the kernel condition is necessary. The condition is satisfied in particular if the perfect kernel of $E \cap \mathbb{T}$ is empty, which means that $E \cap \mathbb{T}$ is countable, and also if P_E has vanishing area measure and no component $C_E(\zeta)$ has positive area measure.

References

[1] F. Bayart, *Boundary behavior and Cesàro means of universal Taylor series*, Rev. Mat. Complut. **19** (2006), 235-247.

[2] F. Bayart, K.-G. Grosse-Erdmann, V. Nestoridis and C. Papadimitropoulos, *Abstract theory of universal series and applications*, Proc. Lond. Math. Soc. **96** (2008), 417-463.

[3] H.-P. Beise, J. Müller, *Generic boundary behaviour of Taylor series in Hardy and Bergman spaces*, Math. Z, **284** (2016), 1185-1197.

[4] L. Bernal-González, A. Jung and J. Müller, *Banach spaces of universal Taylor series in the disc algebra*, Integr. Equ. Oper. Theory, **86** (2016), 1-11.

[5] J.A. Cima, W.T. Ross, *The backward shift on the Hardy space*, American Mathematical Society, Providence, RI, 2000.

[6] G. Costakis, A. Jung, J. Müller, *Generic behavior of classes of Taylor series outside the unit disk*, Constr. Approx. **49** (2019), 509-524.

[7] P. Duren, *Theory of H^p spaces*, Dover Publications, 2000.

[8] O. El-Fallah, K. Kellay, J. Mashreghi, T. Ransford, *A Primer on the Dirichlet Space*, Cambridge University Press, 2014.

[9] S. Gardiner, *Taylor series, universality and potential theory.* In: New Trends in Approximation Theory, Fields Institute Communications, **81** (2018), 247-264; Editors: Javad Mashreghi, Myrto Manolaki, Paul M. Gauthier

[10] K.G. Grosse-Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. 36 (1999), 345-381.

[11] K.G. Grosse-Erdmann, A. Peris Manguillot, *Linear Chaos*, Springer, London, 2011.

[12] V.P. Havin, *Analytic representation of linear functionals in spaces of harmonic and analytic functions continuous in a closed region* Dokl. Akad. Nauk SSSR, **151** (1963) 505-508, English transl. in Soviet Math. Dokl. **4** (1963).

[13] G. Herzog, P. Kunstmann, *Universally divergent Fourier series via Landau's extremal functions*, Comment. Math. Univ. Carolin. **56** (2015), 159-168.

[14] S.V. Hruščev, *The problem of simultaneous approximation and removal of singularities of Cauchy type integrals.* (Russian) Spectral theory of functions and operators. Trudy Mat. Inst. Steklov. **130** (1978), 124-195, English transl. in Proceedings of the Steklov Institute of Mathematics **4** (1979).

[15] S.V. Hruščev, V. Peller, *Hankel operators, best approximation, and stationary Gaussian processes*, Russian Math. Surveys **67** (1982), 61-144.

[16] J. P. Kahane, *Baire's category theorem and trigonometric series*, J. Anal. Math. **80** (2000), 143-182.

[17] A.S. Kechris, *Classical Descriptive Set Theory*, Springer, New York, 1995.

[18] S. Khrushchev, *A continuous function with universal Fourier series on a given closed set of Lebesgue measure zero*, J. Approx. Theory (2020). doi: https://doi.org/10.1016/j.jat.2019.105361.

[19] P. Koosis, *A theorem of Khrushchev and Peller on restrictions of analytic functions having finite Dirichlet integral to closed subsets of the unit circumference.* In: Conference on Harmonic Analysis in honor of Antoni Zygmund, Vol II, Wadsworth, Belmont, 1983.

[20] J. Korevaar, *Tauberian Theory – A Century of Developments*, Springer, Berlin, 2004.

[21] J. Müller, *Spurious limit functions of Taylor series*, Anal. Math. Phys. **9** (2019) 875-885.

[22] V. Nestoridis, *Universal Taylor series*, Ann. Inst. Fourier (Grenoble) **46** (1996), 1293-1306.

[23] S. Shkarin, *Pointwise universal trigonometric series*, J. Math. Anal. Appl. **360** (2009), 754-758.