ℓ^p-type Dirichlet Spaces

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Abstract

In this paper we consider a class of Banach spaces B_p extending the classical Dirichlet space through the growth behaviour of the Taylor coefficients. We focus on the boundary behaviour of functions in B_p and of the sequence of partial sums of their Taylor series.

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1 Introduction and preliminaries

Let $\mathbb{D}$, $\mathbb{T}$ and $\mathbb{C}$ denote the open unit disc, its boundary and the complex plane, respectively. We will write $f \in H(\mathbb{D})$ for an analytic function in $\mathbb{D}$, so that we can represent

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Given $f \in H(\mathbb{D})$, it is said to belong to the classical Dirichlet space D if its Dirichlet integral is finite, that is

$$D := \{f \in H(\mathbb{D}) \, : \, \int_{\mathbb{D}} |f'|^2 \, dm_2 < \infty\},$$

where dm_2 denotes integration with respect to the normalized Lebesgue area measure on $\mathbb{D}$. From $f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}$ it is easily seen that

$$\int_{\mathbb{D}} |f'|^2 \, dm_2 = \sum_{k=1}^{\infty} k |a_k|^2,$$

which implies, in particular, that D is a subspace of the Hardy space H^2 (see [11] for Hardy spaces). The Dirichlet space turns into a Banach space by considering

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the norm

$$\|f\|_D := \left(|f(0)|^2 + \int_{\mathbb{D}} |f'|^2 dm_2\right)^{1/2} = \left(|a_0|^2 + \sum_{k=1}^{\infty} k|a_k|^2\right)^{1/2},$$

which is induced by the scalar product $\langle f, g\rangle := f(0)\overline{g(0)} + \int_{\mathbb{D}} f'\overline{g'}\, dm_2$.

The Dirichlet space has attracted much attention in the last decades. Recommended introductions are the monography [13] and the expository article [27]. It can be actually shown that D is contained in all Hardy spaces H^r, for $r < \infty$, and it turns out that the situation concerning the boundary behaviour of $f \in D$ and, accordingly, of the partial sums $S_n f$ of the Taylor series, is significantly more favourable than in the case of the Hardy spaces: By Beurling's theorem (see e.g. [13] or [27]), the non-tangential limit function of f exists quasi everywhere, that is, up to a set of vanishing (outer) logarithmic capacity, and, by Abel's theorem and Fejér's Tauberian theorem (see e.g. [22], [20, Remarks I.5.5]), the partial sums $S_n f$ converge exactly in the points ζ on the unit circle $\mathbb{T}$ where the non-tangential limit exists. This implies, in particular, that the sequence $(S_n f)_n$ converges to the non-tangential limit function quasi everywhere.

Let now $1 < p \leq \infty$. Several ways of extending the Hilbert space case D to more general L^p-type Banach spaces cases are quite natural.

On the one hand, extending the definition via the area integral leads to the analytic Besov spaces

$$B^p := \{f \in H(\mathbb{D}) : \varphi f' \in L^p(\mathbb{D}, \tau)\},$$

with $\varphi(z) := 1 - |z|^2$ and $d\tau := \varphi^{-2} dm_2$, completely normed by

$$||f||_{B^p} := \left(|f(0)|^p + ||\varphi f'||_{L^p(\mathbb{D},\tau)}\right)^{1/p}$$

(see e.g. [30], [33]). It can be shown that $f \in B^p$ if and only if $\varphi f'' \in L^p(\mathbb{D}, \varphi^{-1} m_2)$ (see e.g. [3, Example 5, p. 18]). With that in mind,

$$B^1 := \{f \in H(\mathbb{D}) : \int_{\mathbb{D}} |f''|\, dm_2 < \infty\}$$

extends the family $(B^p)_{p>1}$ in a natural way. According to [2], the Besov spaces B^p are increasing in p, with B^∞ being the classical Bloch space.

On the other hand, extending the characterisation of the Dirichlet space via convergence of the series $\sum_{k=1}^{\infty} k|a_k|^2$ leads to considering, here for $1 \leq p \leq \infty$, the ℓ^p-type spaces

$$B_p := \{f \in H(\mathbb{D}) : (ka_k)_{k\in\mathbb{N}} \in \ell^p(\mathbb{N}, \nu)\},$$

where $\varphi(k) = k$ and $d\nu := \varphi^{-1} d\mu$ with μ denoting the counting measure on $\mathbb{N}$. For $1 \leq p < \infty$ we have

$$B_p = \{f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{D}) \ : \ \sum_{k=1}^{\infty} k^{p-1} |a_k|^p < +\infty\}$$

with the complete norm

$$\|f\|_{B_p} := \left(|a_0|^p + \sum_{k=1}^{\infty} k^{p-1} |a_k|^p \right)^{1/p}.$$

With these notations, $D = B^2 = B_2$. We also note that B_∞ is the space of all $f \in H(\mathbb{D})$ with $a_k = O(1/k)$ (normed by $||f||_{B_\infty} := |a_0| + \sup_k k|a_k|$) and that B_1 is isomorphic to the analytic Wiener algebra.

For $f \in H(\mathbb{D})$, let $(S_n f)(z) := \sum_{k=0}^{n} a_k z^k$ denote the n-th partial sum of the Taylor series $f(z) = \sum_{k=0}^{\infty} a_k z^k$. From the definition of $\| \cdot \|_{B_p}$ it follows that, for $f \in B_p$, the partial sums are norm-convergent to f for $1 \leq p < \infty$. In particular, the polynomials are dense in B_p.

While the Besov spaces B^p are quite well understood, less is known about the spaces B_p, for $p > 1$. The aim of this paper is to study the boundary behaviour of functions $f \in B_p$ and of the corresponding sequences of partial sums $(S_n f)_n$ on $\mathbb{T}$ (see Sections 2 and 3). Before that, we investigate several basic properties of the spaces B_p.

Note that functions in B_1 extend continuously to $\overline{\mathbb{D}}$. On the other hand,

$$f(z) = \sum_{k=2}^{\infty} \frac{1}{k \log(k)} z^k \quad (z \in \mathbb{D})$$

belongs to B_p for all $p > 1$ and

$$\liminf_{r \to 1^-} f(r) \geq \sum_{k=2}^{\infty} \frac{1}{k \log(k)} = \infty.$$

In particular, f is unbounded in $\mathbb{D}$, that is, f does not belong to H^∞. According to the prime number theorem, the same holds for $f(z) = \sum_{k=1}^{\infty} z^k / p_k$, where p_k denotes the k-th prime number.

Let in the sequel q always denote the conjugate exponent of p, that is

$$pq = p + q.$$

As a consequence of Hölder's inequality and the Hausdorff-Young theorem we get

Proposition 1.1. *If $f(z) = \sum_{k=0}^{\infty} a_k z^k \in B_p$, for some p, then $(a_k)_k \in \ell^s$ for all $s > 1$, and $f \in \bigcap_{r < \infty} H^r$.*

Proof. We may assume that $1 < p < \infty$. For any $p' > q$ (or, equivalently $q' < p$) we have that

$$\sum_{k=1}^{\infty} |a_k|^{q'} = \sum_{k=1}^{\infty} k^{q'/q} \frac{1}{k^{q'/q}} |a_k|^{q'} \leq \left(\sum_{k=1}^{\infty} k^{p-1} |a_k|^p \right)^{q'/p} \left(\sum_{k=1}^{\infty} \frac{1}{k^{\frac{pq'}{q(p-q')}}} \right)^{\frac{p-q'}{p}}$$

A calculation shows that the exponent of the second series being greater than 1 is equivalent to $q' > 1$, and so we obtain the convergence of the geometric series on the right hand side. Now, $f \in B_p$ implies the convergence of the series on the left hand side. With that, the Hausdorff-Young theorem ([12, Theorem A, p. 76]) allows us to conclude that $f \in H^r$ for all $r < \infty$. □

As mentioned above, the Besov spaces are increasing in p. In contrast, the spaces B_p are neither increasing nor decreasing:

Remark 1.2. Let $1 < p < p' \leq \infty$. On the one hand, for $0 < \alpha < \infty$, the function

$$f_\alpha(z) = \sum_{k=2}^{\infty} \frac{1}{k \log^\alpha(k)} z^k$$

belongs to B_p if and only if $\alpha > 1/p$. If we choose $1/p' < \alpha < 1/p$, we obtain that $f_\alpha \in B_{p'}$ but $f_\alpha \notin B_p$. In particular, the spaces B_p are not decreasing in p. On the other hand, let $r, s \in \mathbb{N}$ be so that $q' \leq s/r < q$. A simple calculation yields that the function $f_{r,s}$ given by the lacunary series

$$f_{r,s}(z) = \sum_{k=0}^{\infty} a_k z^k \quad (z \in \mathbb{D}),$$

where $a_k = 1/2^{j \cdot r}$ if $k = 2^{j \cdot s}$ for some $j \in \mathbb{N}$, and zero otherwise, belongs to B_p but not to $B_{p'}$. In particular, the spaces B_p are neither increasing in p. Moreover, if $1 < p < 2$, by choosing $p' = 2$ it is seen that $f_{r,s}$ does not belong to B^t for any $t < 2$, since otherwise, by Theorems A and C from [32], we would have $\sum_{k=1}^{\infty} k|a_k|^t < \infty$, and thus $f_{r,s}$ would also belong to $\bigcap_{t \leq u \leq 2} B_u$. In particular, B_p is not included in $\bigcup_{1<t<2} B^t$.

The functions f_α also show that, for $1 < p \leq \infty$, the space B_p with pointwise multiplication of functions is not an algebra: By choosing $1/p < \alpha < (1+1/p)/2$ we have $(f_\alpha)^2(z) = \sum_{k=4}^{\infty} c_k z^k$ where the coefficients c_k are given by

$$\begin{aligned} c_k &= \sum_{j=2}^{k-2} \frac{1}{j \log^\alpha(j)(k-j) \log^\alpha(k-j)} \\ &\geq \frac{1}{k \log^\alpha(k)} \sum_{j=2}^{k-2} \frac{1}{j \log^\alpha(j)} \geq \frac{1}{k \log^\alpha(k)} \cdot \frac{C}{\log^{\alpha-1}(k)} = \frac{C}{k \log^{2\alpha-1}(k)}. \end{aligned}$$

Hence, we obtain that

$$\sum_{k=4}^{\infty} k^{p-1}|c_k|^p \geq C^p \sum_{k=4}^{\infty} \frac{1}{k \log^{p(2\alpha-1)}(k)} > \sum_{k=4}^{\infty} \frac{1}{k\log(k)} = \infty.$$

For $f, g \in H(\mathbb{D})$ with $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ for all $z \in \mathbb{D}$, the Hadamard product $f * g$ is defined by

$$(f * g)(z) := \sum_{k=0}^{\infty} a_k b_k z^k \quad (z \in \mathbb{D}).$$

With respect to the Hadamard product, B_p becomes an algebra. Actually, more generally we have that $f * g \in B_p$ if $f \in B_p$ and $(b_k)_k$ is bounded. Moreover, from the definition it turns out that

$$B_{2p} = \{f \in H(\mathbb{D}) \,:\, f * f' = (f * f)' \in B_p\}$$

for $p < \infty$.

Using results of Zhu for the Besov spaces we show:

Theorem 1.3. *For $1 \leq p \leq 2$ the space B^p is continuously embedded in B_p and, conversely, for $2 \leq p \leq \infty$ the space B_p is continuously embedded in B^p.*

Proof. Consider the linear mapping $T : B^1 + B^2 = B^2 \to \ell^2(\mathbb{N}, \nu)$ given by $Tf = (a_k)_k$, where $f(z) = \sum_{k=0}^{\infty} a_k z^k$. From Theorem C in [30] it follows that $B^1 \subset B_1$ with continuous inclusion map. Hence, $T|_{B^1}$ maps B^1 continuously into $\ell^1(\mathbb{N}, \nu)$. Since $B^2 = B_2$ with norm equivalence, an application of the complex interpolation theorem (see [33, Theorem 1.32] or [5]) together with Theorem 6.12 in [33] shows that $B^p \subset B_p$ for $1 < p < 2$, with continuous inclusion map.

Now, if $(b_k)_k \in \ell^\infty(\mathbb{N}, \nu)$, that is $(b_k)_k$ is bounded, we have

$$\left|\sum_{k=0}^{\infty} b_k z^k\right| \leq \sup_k |b_k| \frac{1}{1 - |z|}$$

and so $\varphi g \in L^\infty(\mathbb{D}, \tau)$, where $g(z) = \sum_{k=0}^{\infty} b_k z^k$. Also, g belongs to the Bergman space

$$A^2 = \{g \in H(\mathbb{D}) : \int_{\mathbb{D}} |g|^2 \, dm_2 < \infty\}$$

if and only if $(b_k)_k \in \ell^2(\mathbb{N}, \nu)$. This shows that $T((b_k)_k) := \varphi g$ defines a (bounded) linear mapping $T : \ell^2(\mathbb{N}, \nu) + \ell^\infty(\mathbb{N}, \nu) \to L^2(\mathbb{D}, \tau) + L^\infty(\mathbb{D}, \tau)$. An application of the Riesz-Thorin interpolation theorem shows that T maps $\ell^p(\mathbb{N}, \nu)$ boundedly to $L^p(\mathbb{D}, \tau)$ for $2 < p < \infty$. Now, if $f \in B_p$, then $(b_k)_k = (ka_k)_k \in \ell^p(\mathbb{N}, \nu)$, and so $\varphi f'$ belongs to $L^p(\mathbb{D}, \tau)$, which means that $f \in B^p$. □

2 Growth and boundary behaviour

Note that functions in B_∞ belong to the Bloch space B^∞, which means that

$$f(z) = O\left(\log\left(\frac{1}{1-|z|}\right)\right) \quad (|z| \to 1^-)$$

for $f \in B_\infty$. We shall prove that for functions in B_p the growth is restricted by $\log^{1/q}(1/(1-|z|^q))$ (cf. [13, Theorem 1.2.1] for the case $p=2$). To this aim, for each $w \in \mathbb{D}$, we compute the norm of the evaluation functional $\Lambda_w : B_p \to \mathbb{C}$ given by $\Lambda_w f := f(w)$.

Note first that

$$\langle f, g\rangle := a_0\overline{b_0} + \sum_{k=1}^{\infty} k a_k \overline{b_k}, \tag{1}$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k \in B_p$, $g(z) = \sum_{k=0}^{\infty} b_k z^k$, defines a linear-antilinear pairing for the spaces B_p and B_q. Indeed, by the Hölder-Young inequality, and writing $k = k^{1/p}k^{1/q}$, we obtain that

$$|a_0\overline{b_0}| + \sum_{k=1}^{\infty} k|a_k\overline{b_k}| \le \left(|a_0|^p + \sum_{k=1}^{\infty} k^{p-1}|a_k|^p\right)^{1/p} \left(|b_0|^q + \sum_{k=1}^{\infty} k^{q-1}|b_k|^q\right)^{1/q}$$

Since $(kz^{k-1})_{k\in\mathbb{N}}$ is an orthonormal system in $L^2(\mathbb{D}, m_2)$, it is easily seen that

$$\langle f - a_0, g - b_0\rangle = \int_{\mathbb{D}} f'\overline{g'}\, dm_2$$

(cf. [8, Proposition 6.4.2], [2]).

In particular, $\phi_g(f) := \langle f, g\rangle$ defines a bounded linear functional on B_p, that is, $\phi_g \in (B_p)'$, with $\|\phi_g\|_{(B_p)'} \le \|g\|_{B_q}$. Actually, every functional of $(B_p)'$ admits such a representation:

Proposition 2.1. *Let $1 < p < \infty$. Then $g \mapsto \phi_g$ maps B_q isometrically isomorphic to $(B_p)'$.*

Proof. According to the preliminary considerations, it suffices to show that each $\phi \in (B_p)'$ is of the form ϕ_g and that $\|g\|_{B_q} \le \|\phi\|_{(B_p)'}$. So let $\phi \in (B_p)'$ be given and let $g(z) := \sum_{k=0}^{\infty} b_k z^k$ where $b_0 := \phi(1)$ and $b_k := \phi(z^k)/k$ for $k \in \mathbb{N}$. Now, consider the sequence $(c_k)_k$ defined by

$$\begin{aligned} c_0 &:= |b_0|^{q-2}\overline{b_0}, \\ c_k &:= k^{q-2}|b_k|^{q-2}\overline{b_k}, \quad (k \in \mathbb{N}), \end{aligned}$$

with $|b_k|^{q-2}\overline{b_k} := 0$ if $b_k = 0$. Then, we have that $c_0 b_0 = |b_0|^q$ and $c_k b_k = k^{q-2}|b_k|^q$ $(k \in \mathbb{N})$, while on the other hand $|c_0|^p = |b_0|^q$ and $k^{p-1}|c_k|^p =$

$k^{q-1}|b_k|^q$ $(k \in \mathbb{N})$. If we fix an arbitrary $N \in \mathbb{N}$, from the boundedness of ϕ we obtain that

$$|b_0|^q + \sum_{k=1}^{N} k^{q-1}|b_k|^q = \phi\left(\sum_{k=0}^{N} c_k z^k\right) \le \|\phi\|_{(B_p)'} \left(|c_0|^p + \sum_{k=1}^{N} k^{p-1}|c_k|^p\right)^{1/p}.$$

Putting all together we obtain that

$$\left(|b_0|^q + \sum_{k=1}^{N} k^{q-1}|b_k|^q\right)^{1/q} \le \|\phi\|_{(B_p)'}.$$

Finally, letting $N \to \infty$ gives us $\|g\|_{B_q} \le \|\phi\|_{(B_p)'}$, and from the definition of $(b_k)_k$ we have $\phi = \phi_g$. □

Now, for $w \in \mathbb{D}$ we consider the function $k_w \in H(|w|^{-1}\mathbb{D})$ given by

$$k_w(z) := 1 + \log\left(\frac{1}{1 - \overline{w}z}\right) = 1 + \sum_{k=1}^{\infty} \frac{\overline{w}^k}{k} z^k.$$

Then

$$\|k_w\|_{B_q}^q = 1 + \log\left(\frac{1}{1 - |w|^q}\right) = \log\left(\frac{e}{1 - |w|^q}\right),$$

and for $f(z) = \sum_{k=0}^{\infty} a_k z^k$ we have

$$\Lambda_w f = a_0 + \sum_{k=1}^{\infty} k a_k \frac{w^k}{k} = \langle f, k_w \rangle.$$

So, we can view the functions $k_w \in B_q$ as a kind of reproducing kernel in B_p. From Proposition 2.1 we obtain

$$\|\Lambda_w\|_{(B_p)'} = \|k_w\|_{B_q} = \log^{1/q}\left(\frac{e}{1 - |w|^q}\right),$$

and as a consequence, we have:

Theorem 2.2. *If $1 < p < \infty$ and $f \in B_p$, then*

$$|f(z)| \le \log^{1/q}\left(\frac{e}{1 - |z|^q}\right)\|f\|_{B_p} \quad (z \in \mathbb{D}).$$

Remark 2.3. Let $\varepsilon : \mathbb{D} \to (0, \infty)$ be a function such that $\liminf_{|z| \to 1^-} \varepsilon(z) = 0$. Then, there exists $f \in B_p$ such that

$$f(z) \ne O\left(\varepsilon(z) \log^{1/q}\left(\frac{e}{1 - |z|^q}\right)\right) \quad (|z| \to 1^-).$$

Indeed: Let $(w_n)_n$ be a sequence in $\mathbb{D}$ with $\varepsilon(w_n) \to 0$. Consider the sequence of functions $(g_n)_n$ in B_q given by

$$g_n(z) := \varepsilon(w_n)^{-1} \log^{-1/q}\left(\frac{e}{1-|w_n|^q}\right) k_{w_n}(z) \quad (z \in \mathbb{D}).$$

Since

$$\|g_n\|_{B_q} = \varepsilon(w_n)^{-1} \log^{-1/q}\left(\frac{e}{1-|w_n|^q}\right) \|k_{w_n}\|_{B_q} = \varepsilon(w_n)^{-1} \to \infty$$

as $n \to \infty$, the sequence $(g_n)_n$ is unbounded in B_q. By the Banach-Steinhaus theorem, there exists $f \in B_p$ such that $\sup_{n\geq 1} |\langle f, g_n\rangle| = \infty$.

In the sequel we investigate the boundary functions and the behaviour of the partial sums $S_n f$ of B_p-functions. We start with an extension of Fejér's Tauberian theorem mentioned in the introduction. It is formulated in [20, Remark 5.5] with the comment that the proof follows along the same lines as the proof of Fejér's theorem. Since it is basic for our purposes, we include a proof. For $p = \infty$ the result also holds, and is the classical Littlewood's theorem (see [34, Vol I, Theorem III 1.38]).

Proposition 2.4. *Let $f \in B_p$, where $1 < p \leq \infty$. Then, the sequence of partial sums of the Taylor series $(S_n f(\zeta))_n$ converges at every point $\zeta \in \mathbb{T}$ at which the radial limit of f exists.*

Proof. Let $p < \infty$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$. We put $\varepsilon_n := \sum_{k=n}^{\infty} k^{p-1}|a_k|^p$ and take $n \in \mathbb{N}$ so that $r_n := 1 - \varepsilon_n^{1/p}/n > 0$. Then, for all $\zeta \in \mathbb{T}$, we have that

$$\begin{aligned} \left|\sum_{k=0}^{n-1} a_k \zeta^k - f(r_n\zeta)\right| &= \left|\sum_{k=0}^{n-1} a_k \zeta^k (1-r_n^k) - \sum_{k=n}^{\infty} a_k r_n^k \zeta^k\right| \\ &\leq (1-r_n)\sum_{k=0}^{n-1} k|a_k| + \sum_{k=n}^{\infty} |a_k| r_n^k \end{aligned}$$

Applying the Hölder inequality and $k = k^{1/p}k^{1/q}$ gives

$$\begin{aligned} (1-r_n)\sum_{k=0}^{n-1} k|a_k| &\leq (1-r_n)\left(\sum_{k=0}^{n-1} k^{p-1}|a_k|^p\right)^{1/p} \left(\sum_{k=0}^{n-1} k^{q/p}\right)^{1/q} \\ &\leq (1-r_n)\varepsilon_0^{1/p} n = \varepsilon_0^{1/p}\varepsilon_n^{1/p} \to 0 \quad (n\to\infty) \end{aligned}$$

and

$$\begin{aligned} \sum_{k=n}^{\infty} |a_k| r_n^k &\leq \frac{1}{n^{1/q}} \sum_{k=n}^{\infty} k^{1/q}|a_k| r_n^k \leq \frac{1}{n^{1/q}} \left(\sum_{k=n}^{\infty} k^{p-1}|a_k|^p\right)^{1/p} \left(\sum_{k=n}^{\infty} r_n^{kq}\right)^{1/q} \\ &\leq \frac{1}{n^{1/q}} \varepsilon_n^{1/p} \frac{1}{(1-r_n)^{1/q}} = \varepsilon_n^{1/p^2} \to 0 \quad (n\to\infty) \end{aligned}$$

□

In combination with Abel's limit theorem, the above Tauberian result shows that, for functions in B_p, convergence of the partials sum $(S_n f)(\zeta)$ and existence of a radial limit of f at ζ are equivalent. In order to get information about sets of convergence on $\mathbb{T}$ we relate the spaces B_p (and B^p) to other Banach spaces of holomorphic functions in the disc.

It is well-known that, for $1 < p < \infty$, functions in H^p are the Cauchy integral of their boundary function belonging to $L^p(\mathbb{T}, m_1)$, with m_1 denoting the arc length measure on $\mathbb{T}$. For $p > 1$ and $0 < \beta < 1$, the space H^p_β is the space of all $f \in H(\mathbb{D})$ for which there exists $F \in L^p(\mathbb{T}, m_1)$ such that

$$f(z) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{F(\zeta)}{(1 - z\bar{\zeta})^{1-\beta}} \, dm_1(\zeta), \quad z \in \mathbb{D}.$$

The boundary behaviour of functions in the latter spaces was studied in [19] and [26]. By considering an arbitrary exponent α of the weight function φ, the class B^p can be extended into the more general Dirichlet-type spaces D^p_α, defined by

$$D^p_\alpha := \{f \in H(\mathbb{D}) \ : \ \int_{\mathbb{D}} \varphi^\alpha |f'|^p \, dm_2 < \infty\}$$

for $p > 1$ and $\alpha \in \mathbb{R}$. In particular, we have $B^p = D^p_{p-2}$ for $1 < p < \infty$. The spaces D^p_α were studied e.g. in [14], [28], [29]. The approach in these papers is to represent functions in D^p_α through the class H^p_β. Among others, Girela and Peláez ([14]) showed that the inclusion

$$D^p_\alpha \subset H^p_{(p-\alpha-1)/p}$$

holds true whenever $-1 < \alpha < p - 1$ and $1 < p \leq 2$, and the converse inclusion was proved by Twomey (see [28]) if $p \geq 2$. So, in particular, $B^p \subset H^p_{1/p}$ for $1 < p \leq 2$. In [29] also the spaces B_p are considered. It is shown that $B_p \subset H^q_{1/q}$ for $1 < p \leq 2$, while $B_p \subset H^p_{1/p}$ for $p \geq 2$.

Let $C_{\alpha,p}$ denote the Bessel capacity (see [23], [1]; cf. [29]). The capacities $C_{1/p,p}$ are ordered in the sense that $C_{1/r,r}(E) = 0$ implies $C_{1/s,s}(E) = 0$ for $1 < r < s < \infty$ (see [23], cf. [29]). Moreover, $C_{1/2,2}$-capacity is equivalent to logarithmic capacity in the sense that $C_{1/2,2}(E) = 0$ if and only if the logarithmic capacity of E vanishes. Thus, in particular, if $C_{1/r,r}(E) = 0$ for some $1 < r < 2$, then the logarithmic capacity of E vanishes.

Remark 2.5. As a consequence of [29, Theorem 1 and Lemma], it follows that, for $1 < p \leq 2$ and for any $f \in B_p$, the sequence $(S_n f)_n$ converges $C_{1/q,q}$-quasi everywhere on $\mathbb{T}$, and, for any $f \in B^p$, convergence holds $C_{1/p,p}$-quasi everywhere. Moreover, if $p \geq 2$, then $C_{1/p,p}$-quasi everywhere convergence of the sequence $(S_n f)_n$ holds for all $f \in B_p$.

From Theorem 2.4 and the fact that Cesaro summability at $\zeta \in \mathbb{T}$ implies the existence of the non-tangential limit at ζ (see [34, Vol I, Theorem III 1.34]) we finally obtain:

Theorem 2.6. *For $1 < p \leq \infty$, $f \in B_p$ and $\zeta \in \mathbb{T}$ the following statements are equivalent:*

1. $(S_n f(\zeta))_n$ *converges.*
2. $(S_n f(\zeta))_n$ *is Cesaro summable.*
3. f *has a non-tangential limit at* ζ.
4. f *has a radial limit at* ζ.

The conditions hold $C_{1/q,q}$-quasi everywhere for $1 < p \leq 2$ and $C_{1/p,p}$-quasi everywhere for $2 < p < \infty$.

3 Sets of universality

In the last decades, universality properties of various forms have been investigated. We consider universality of the sequence of partial sums $S_n f$. For f holomorphic in $\mathbb{D}$, Λ an infinite subset of $\mathbb{N}_0$ and E a closed subset of $\mathbb{T}$ we say that the sequence of partial sums $(S_n f)_{n\in\Lambda}$ is universal, if $\{S_n f : n \in \Lambda\}$ is a dense set in $C(E)$, where $C(E)$ denotes the space of all continuous functions on E endowed with the uniform norm. For X a Banach space of functions holomorphic in $\mathbb{D}$, we call the closed set $E \subset \mathbb{T}$ a set of universality for X if for all infinite sets $\Lambda \subset \mathbb{N}_0$ a residual set of functions in X exists with the property that $(S_n f)_{n\in\Lambda}$ is universal on E.

In [4] it was proved that each closed set of vanishing arc length measure is a set of universality for all Hardy spaces H^p, where $p < \infty$. According to Twomey's results (Remark 2.5), this cannot be the case for any of the spaces B_p, where $p < \infty$, or B^p with $p \leq 2$. Khrushchev ([17, Theorem 3.2]) recently showed that, for each closed $E \subset \mathbb{T}$ with $\mathrm{cap}_p(E) = 0$, where the capacity cap_p is determined by an appropriate Besov space norm (see also [18, p. 124]), there are functions in the Besov space B^p so that $(S_n f)_{n\in\mathbb{N}}$ is universal on E. Since $\mathrm{cap}_p(E) = 0$ if and only if the logarithmic capacity of E vanishes, this shows in particular that functions in the Dirichlet space D with universal Taylor series on E exist.

The universality result turns out to be a consequence of a result on simultaneous approximation by polynomials. We will show that a similar approximation result holds for B_p on appropriate small closed sets $E \subset \mathbb{T}$, and with that we also prove the existence of universal Taylor series.

Remark 3.1. If $F, G \subset \mathbb{T}$ are closed sets, then the product set $F \cdot G := \{z_1 \cdot z_2 : z_1 \in F, z_2 \in G\}$ is easily seen to be also closed in $\mathbb{T}$. In particular, if $E \subset \mathbb{T}$ is closed, then the product set $E^d := \{z_1 \cdots z_d : z_1, \ldots, z_d \in E\}$ ($d \in \mathbb{N}$) of E is also closed in $\mathbb{T}$. On the other hand, if $F, G \subset \mathbb{T}$ are closed sets with logarithmic capacity zero, this does not imply that the product set $F \cdot G$ has also logarithmic capacity zero (see [24, Section 6]).

We write $p_0 := \infty$ and $p_d := 2d/(2d-1)$ for $d \in \mathbb{N}$.

Theorem 3.2. *Let $d \in \mathbb{N}$ and $p_d \leq p < p_{d-1}$. Then, each closed set $E \subset \mathbb{T}$ so that E^d has logarithmic capacity zero is a set of universality for B_p.*

As a consequence of Theorem 3.2 and the Tauberian theorem 2.4, we obtain the following extension of the converse of Beurling's theorem for the Dirichlet space due to Carleson (see e.g. [7], [27, Theorem 5.4], and [13, Theorem 3.4.1] for a strengthened version).

Corollary 3.3. *Let $d \in \mathbb{N}$ and $p_d \leq p < p_{d-1}$. If $E \subset \mathbb{T}$ is closed and so that E^d has logarithmic capacity zero, then for a residual set of functions $f \in B_p$ radial limits do not exist in any point of E.*

As formulated in [10, Lemma 2.5] (cf. also the proof of Theorem 1.1 in [4]), an application of the Universality Criterion (see [15] or [16]) shows that, for Theorem 3.2, it suffices to prove the following result on simultaneous approximation by polynomials in B_{p_d} and $C(E)$, where $C(E)$ is endowed with the uniform norm $\|\cdot\|_E$).

Theorem 3.4. *Let $d \in \mathbb{N}$ and $p_d \leq p < p_{d-1}$. If $E \subset \mathbb{T}$ is a closed set such that E^d has logarithmic capacity zero, then for all $(f, g) \in B_p \times C(E)$ and all $\varepsilon > 0$, there is a polynomial P such that $\|f - P\|_{B_p} < \varepsilon$ and $\|g - P\|_{C(E)} < \varepsilon$.*

Remark 3.5. For the Besov spaces B^p a similar result on simultaneous approximation holds for sets with $\mathrm{cap}_p(E) = 0$ (see [17, proof of Theorem 3.2]). Note, however, that, due to the lack of a corresponding Tauberian theorem, in contrast to the case of functions B_p this does not give information on the non-existence of radial limits on sets E with $\mathrm{cap}_p(E) = 0$. For the disc algebra it turns out that E is a set of universality if and only if E is finite (see [6]). Note that here unrestricted limits exist in all points of $\mathbb{T}$. Also, this shows that a simultaneous approximation property as above is not necessary for having universality.

We turn to the proof of the central Theorem 3.4, and start with several notions and preliminary results.

Let $X = (X, ||\cdot||_X)$ be a Banach space of holomorphic functions on $\mathbb{D}$ or of continuous functions on a subset of $\mathbb{T}$ so that the polynomials are dense in X, and that

$$r_X := \limsup_{n\to\infty} ||P_n||_X^{1/n} < \infty$$

with $P_n(z) := z^n$. In this case we will say that X is regular. In particular, regular spaces are separable since the polynomials with (Gaußian) rational coefficients also form a dense subset. By X' we denote the norm dual of X, that is, the space of bounded linear functionals on X, and by $H(0)$ the linear space of germs of functions holomorphic at 0. Then, the Cauchy transform $C_X : X' \to H(0)$ with respect to X is defined by

$$(C\phi)(w) := (C_X\phi)(w) = \sum_{k=0}^{\infty} \phi(P_k) w^k$$

for $|w| < 1/r_X$ and $\phi \in X'$. Since the polynomials form a dense set in X, the Hahn-Banach theorem implies that C_X is injective. By definition, the range R_X of C_X is the Cauchy dual of X. For closed $E \subset \mathbb{T}$, the norm dual of $C(E)$ is the space of Borel measures supported on E (with the total variation norm), and the Cauchy dual is the set of all restrictions to $\mathbb{D}$ of Cauchy integrals

$$\widehat{\mu}(w) := \int \frac{1}{1 - w\overline{\zeta}} \, d\mu(\zeta) \qquad (w \in \mathbb{C} \setminus E)$$

of a complex Borel measure with support in E.

The following consequence of the Hahn-Banach theorem (see [18, Theorem 1.2], [10, Lemma 2.7]) is the basis for our subsequent considerations.

Lemma 3.6. *Let X and Y be regular. Then, $R_X \cap R_Y = \{0\}$ if and only if the pairs (P, P), where P ranges over the set of polynomials, form a dense set in the sum $X \oplus Y$.*

Remark 3.7. Using Lemma 3.6, the statement on simultaneous approximation from Theorem 3.4 can be transformed into an equivalent one saying that no non-zero function in the Cauchy dual of B_{p_d} can coincide on $\mathbb{D}$ with some Cauchy transform $\widehat{\mu}$ for a measure μ supported on E (cf. [4, Lemma 2.1]).

For $p > 1$ and $\gamma \in \mathbb{R}$ we consider the two parameter family of spaces $B_{p,\gamma}$, given by

$$B_{p,\gamma} := \{f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{D}) \ : \ \sum_{k=1}^{\infty} k^\gamma |a_k|^p < +\infty\},$$

which become Banach spaces when endowed with the norm

$$\|f\|_{B_{p,\gamma}} := \left(|a_0|^p + \sum_{k=1}^{\infty} k^\gamma |a_k|^p \right)^{1/p}.$$

In particular, $B_{2,-1}$ is the classical Bergman space A^2.

Hölder's inequality shows that for $\gamma > p - 1$ the spaces $B_{p,\gamma}$ are contained in the analytic Wiener algebra. In [29], results on the convergence of $(S_n f)$ for

$0 < \gamma \le p-1$ and functions $f \in B_{p,\gamma}$ were obtained by relating the spaces $B_{p,\gamma}$ to appropriate H^p_β. In the limiting case $B_p = B_{p,p-1}$, Theorem 3.2 provides a result in the converse direction. The proof relies mainly on the fact that the Cauchy dual is of a certain Bergman type:

Proposition 3.8. *Let $\gamma \in \mathbb{R}$ and $1 < p < \infty$. Then, the Cauchy dual of $B_{p,\gamma}$ equals $B_{1,-\gamma q/p}$ with $||\phi||_{(B_{p,\gamma})'} = ||C\phi||_{B_{q,-\gamma q/p}}$ for each $\phi \in (B_{p,\gamma})'$. In particular, the Cauchy dual of B_p is $B_{q,-1}$.*

Proof. Given $g(w) = \sum_{k=0}^\infty b_k w^k \in B_{q,-\gamma q/p}$ and $f(z) = \sum_{k=0}^\infty a_k z^k \in B_{p,\gamma}$, Hölder's inequality yields

$$\sum_{k=0}^\infty |a_k b_k| = |a_0 b_0| + \sum_{k=1}^\infty |k^{\gamma/p} a_k k^{-\gamma/p} b_k| \le \|f\|_{B_{p,\gamma}} \|g\|_{B_{q,-\gamma q/p}}.$$

Hence, $\phi_g(f) := \sum_{k=0}^\infty a_k b_k$ defines a bounded linear functional on $B_{p,\gamma}$ with $\|\phi_g\|_{(B_{p,\gamma})'} \le \|g\|_{B_{q,-\gamma q/p}}$, and $C\phi_g = g$.

On the other hand, for $\phi \in (B_{p,\gamma})'$ and $k \in \mathbb{N}$, let $g := C\phi$ be the Cauchy transform of ϕ, and $b_k := \phi(P_k)$. By considering the sequence $(c_k)_k$ defined by $c_0 := |b_0|^{q-2} b_0$ and $c_k := k^{-\gamma q/p} |b_k|^{q-2} b_k$ for $k \in \mathbb{N}$, in a similar way as in the proof of Proposition 2.1 it can be shown that $\|g\|_{B_{q,-\gamma q/p}} \le \|\phi\|_{(B_{p,\gamma})'}$. □

If $f \in H(\mathbb{C}_\infty \setminus E_1)$ and $g \in H(\mathbb{C}_\infty \setminus E_2)$ with E_1, E_2 compact subsets of $\mathbb{T}$ and $\mathbb{C}_\infty$ the extended plane, then $E_1 \cdot E_2$ is compact, and if $E_1 \cdot E_2 \ne \mathbb{T}$, the Hadamard multiplication theorem implies that $f * g \in H(\mathbb{C}_\infty \setminus (E_1 \cdot E_2))$ with

$$(f * g)(z) = \sum_{k=0}^\infty a_{-k} b_{-k} / z^{k+1}$$

in $\mathbb{D}_e = \mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ if $f(z) = \sum_{k=0}^\infty a_{-k}/z^{k+1}$ and $g(z) = \sum_{k=0}^\infty b_{-k}/z^{k+1}$ in $\mathbb{D}_e$ (see [25, Theorem 2.7, Example 2.8]).

Let $d \in \mathbb{N}$, $f \in H(\mathbb{D})$ with $f(z) = \sum_{k=0}^\infty a_k z^k$. We write f^{*d} for the d-times iterated Hadamard product

$$f^{*d}(z) := \sum_{k=0}^\infty a_k^d z^k.$$

With that we have

$$B_{2d,-1} = \{f \in H(\mathbb{D}) \,:\, f^{*d} \in A^2\}.$$

So far we have worked with the spaces $B_{p,\gamma}$ on the unit disc. We need to take into consideration the analogous spaces on the complement of the closed unit disc with respect to $\mathbb{C}_\infty$.

Definition 3.1. Let $\gamma \in \mathbb{R}$ and $1 < p < \infty$. We write $\mathbb{D}_e = \mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ and define $B_{p,\gamma,e}$ as the space of all functions $f(z) = \sum_{k=0}^\infty b_k/z^{k+1} \in H(\mathbb{D}_e)$ such that

$$\|f\|_{B_{p,\gamma,e}}^p := \sum_{k=1}^\infty k^\gamma |b_k|^p < \infty.$$

Moreover, for closed subsets E of $\mathbb{T}$ we write

$$B_{p,\gamma}(\mathbb{C}_\infty \backslash E) := \{f \in H(\mathbb{C}_\infty \backslash E) \ : \ f|_{\mathbb{D}_e} \in B_{p,\gamma,e}, \ f|_{\mathbb{D}} \in B_{p,\gamma}\}.$$

Remark 3.9. A classical theorem on removable singularities for functions in Bergman spaces (see, e.g. [13, p. 178] or [9]) says that $B_{2,-1}(\mathbb{C}_\infty \setminus E)$ reduces to the zero space if E is a closed subset of $\mathbb{T}$ of vanishing logarithmic capacity. Now, if $d \in \mathbb{N}$, according to the Hadamard multiplication theorem, for $f \in B_{2d,-1}(\mathbb{C}_\infty \backslash E)$ we have $f^{*d} \in B_{2,-1}(\mathbb{C}_\infty \setminus E^{*d})$. So, if E is a closed subset of $\mathbb{T}$ so that E^d is of logarithmic capacity zero, then

$$B_{2d,-1}(\mathbb{C}_\infty \backslash E) = \{0\}.$$

We finally highlight a remarkable result of Khrushchev and Peller (Remark after Corollary 3.8 in [18]; see also [21] for a very nice and simple proof).

Lemma 3.10. *Let μ be a complex measure supported on $\mathbb{T}$ and let $d \in \mathbb{N}$. Then $\widehat{\mu}|_{\mathbb{D}} \in B_{2d,-1}$ implies $\widehat{\mu} \in B_{2d,-1}(\mathbb{C}_\infty \setminus \mathbb{T})$*

With that we are in a position to give the proof of Theorem 3.4, and with that in particular of Theorem 3.2:

Proof of Theorem 3.4. For $p_d \leq p < p_{d-1}$ we have $2d - 2 < q \leq 2d$. Let

$$f(z) = \sum_{k=0}^\infty a_k z^k \in B_{q,-1}$$

be so that $f = \widehat{\mu}$ for some complex measure μ supported on E. Then,

$$a_k = \int \overline{\zeta}^k \, d\mu(\zeta)$$

for $k \in \mathbb{N}_0$ and with that $|a_k| \leq |\mu|(F)$ for all k. Since $\sum_{k=0}^\infty |a_k|^q/(k+1) < \infty$, the boundedness of $(a_k)_k$ implies that also $\sum_{k=0}^\infty |a_k|^{2d}/(k+1) < \infty$. Now, Lemma 3.10 shows that $\widehat{\mu}$ belongs to $B_{2d,-1}(\mathbb{C}_\infty \setminus E)$. But then Remark 3.9 implies that $f = 0$. As an application of Lemma 3.6 with $X = B_p$ and $Y = C(E)$, the statement of Theorem 3.4 holds. □

Remark 3.11. Let $E \subset \mathbb{T}$ be closed set having positive logarithmic capacity. Then Beurling's Theorem implies that simultaneous approximation as in Theorem 3.4 does not hold for $D = B_2$, and thus Lemma 3.6 implies the existence of a non-zero function $f \in A^2$ that coincides with the Cauchy transform $\widehat{\mu}$ of some complex measure μ supported on E. The proof of Theorem 3.4 yields then that f also belongs to $B_{q,-1}$, for all $q \geq 2$. Lemma 3.6 now shows that simultaneous approximation as in Theorem 3.4 does not hold for any of the spaces B_p, where $1 < p \leq 2$.

Let $A(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}) : f \text{ holomorphic in } \mathbb{D}\}$ denote the disc algebra, and let $E \subset \mathbb{T}$ be closed. The Rudin-Carleson theorem states that for every $f \in C(E)$ there exists $g \in A(\mathbb{D})$ such that $f = g$ on E if E has arc length measure zero. Khrushchev and Peller proved that a similar result holds for $A(\mathbb{D}) \cap D$ if the logarithmic capacity of E vanishes and, more generally, for $A(\mathbb{D}) \cap B^p$ if $\mathrm{cap}_p(E) = 0$ (see [18, Theorem 3.17], [21], cf. [13, Section 4.3]). According to results of Wallin and Sjödin, the corresponding conditions turn out to be also necessary (see [18], [21]).

The main ingredient for the proof of the Khrushchev-Peller theorem is Theorem 3.8 from [18], which has Lemma 3.10 as corollary. A second important fact is that for complex measures on $\mathbb{T}$ with finite p-energy and closed sets $E \subset \mathbb{T}$ with $\mathrm{cap}_{\mathrm{p}}(E) = 0$ the measure μ vanishes on all closed subsets of E (see [18, Lemma 3.7], cf. [21, Lemma 1]). By observing that μ vanishes on all closed subsets F of E if the d-fold convolution μ^{*d} vanishes on all F^d, and by following and adapting the proof of [18, Theorem 3.17] (or again [21]) one can deduce:

Theorem 3.12. *Let $d \in \mathbb{N}$ and let $E \subset \mathbb{T}$ be a closed set such that E^d has logarithmic capacity zero. Then each function in $C(E)$ is the restriction to E of a function in $A(\mathbb{D}) \cap B_{p_d}$.*

References

[1] D.R. Adams, L.I. Hedberg, *Functions Spaces and Potential Theory*, Springer, Berlin (1996).

[2] J. Arazy, S. Fisher, J. Peetre, *J. Möbius invariant function spaces.* J. Reine Angew. Math. **363**, 110–145 (1985).

[3] J. Arazy, S. Fisher, J. Peetre: Möbius invariant spaces of analytic functions. In: Berenstein C.A. (eds) Complex Analysis I. Lecture Notes in Mathematics, vol 1275. Springer, Berlin, Heidelberg (1987).

[4] H.P. Beise, J. Müller, *Generic boundary behaviour of Taylor series in Hardy and Bergman spaces.* Math. Z. **284**, 1185–1197 (2016).

[5] J. Bergh and J. Löfström: Interpolation Spaces – An Introduction, Springer-Verlag, Berlin, (1976).

[6] L. Bernal-González, A. Jung, J. Müller, *Banach spaces of universal Taylor series in the disc algebra.* Integr. Equ. Oper. Theory **86**, 1–11 (2016).

[7] L. Carleson: Selected problems on exceptional sets. Selected reprints, Wadsworth Math. Ser., Wadsworth, Belmont, CA, (1983).

[8] R. Cheng, J. Mashreghi, W.T. Ross: Function theory and ℓ^p spaces, American Mathematical Society, Providence, RI (2020).

[9] J.B. Conway: Functions of one Complex Variable II. Springer, New York (1995).

[10] G. Costakis, A. Jung, J. Müller, *Generic behavior of classes of Taylor series outside the unit disk*. Constr. Approx. **49**, 509–524 (2019).

[11] P. Duren: Theory of H^p Spaces. Dover Publications (2000).

[12] P. Duren, A. Schuster: Bergman Spaces. Mathematical surveys and monographs, no. 100, American Mathematical Society (2000).

[13] O. El-Fallah, K. Kellay, J. Mashreghi, T. Ransford: A Primer on the Dirichlet Space. Cambridge University Press (2014).

[14] D. Girela, J.A. Peláez, *Boundary behaviour of analytic functions in spaces of Dirichlet type*. J. Inequal. Appl. **2006**, 12 pp. (2006).

[15] K.G. Grosse-Erdmann, *Universal families and hypercyclic operators. Bull. Amer. Math. Soc.* **36**, 345–381 (1999).

[16] K.G. Grosse-Erdmann, A. Peris Manguillot: Linear Chaos. Springer, London (2011).

[17] S.V. Khrushchev, *A continuous function with universal Fourier series on a given closed set of Lebesgue measure zero*, J. Approx. Theory, **252** (2020).

[18] S.V. Khrushchev, V. Peller, *Hankel operators, best approximation, and stationary Gaussian processes*, Russian Math. Surveys **67**, 61–144 (1982)

[19] J.R. Kinney, *Tangential limits of functions of the class S_α*, Proceedings of the American Mathematical Society **14**, 68–70 (1963).

[20] J. Korevaar: Tauberian Theory. Springer, Berlin (2004).

[21] P. Koosis, *A theorem of Khrushchëv and Peller on restrictions of analytic functions having finite Dirichlet integral to closed subsets of the unit circumference.* Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), 740–748, Wadsworth Math. Ser., Wadsworth, Belmont, CA, (1983).

[22] E. Laudau, D. Gaier, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie.* (German) [Presentation and explanation of some more recent results in function theory] 3rd ed. Springer, Berlin (1986).

[23] N.G. Meyers, *A theory of capacities for potentials of functions in Lebesgue classes*, Math. Scand. **26**, 255–292, (1970).

[24] M.A. Monterie, *Capacities of certain Cantor sets*, Indag. Math. (N.S.) **8**, no. 2, 247–266 (1997).

[25] J. Müller, T. Pohlen, *The Hadamard product on open sets in the plane*, Complex Anal. Oper. Theory **6**, 257–274 (2012).

[26] A. Nagel, W. Rudin, and J.H. Shapiro, *Tangential boundary behaviour of functions in Dirichlet-type spaces*, Annals of Mathematics. Second Series **116**, no. 2, 331–360 (1982).

[27] W.T. Ross, *The classical Dirichlet space. Recent advances in operator-related function theory*, 171–197, Contemp. Math. **393**, Amer. Math. Soc. Providence, RI, (2006).

[28] J.B. Twomey, *Boundary Behaviour and Taylor Coefficients of Besov Functions*, Comput. Methods Funct. Theory **14**, no. (2–3), 227–236 (2014).

[29] J.B. Twomey, *The capacity of sets of divergence of certain Taylor series on the unit circle*, Comput. Methods Funct. Theory **19**, no. 2, 227–236 (2019).

[30] K. Zhu, *Analytic Besov spaces*. J. Math. Anal. Appl. **157**, no. 2, 318–336 (1991)

[31] K. Zhu, *Duality of Bloch spaces and norm convergence of Taylor series.* Michigan Math. J. **38**, 89–101 (1991).

[32] K. Zhu, *A class of Mbius invariant function spaces.* Illinois J. Math. **51**, no. 3, 977–1002 (2007).

[33] K. Zhu: Spaces of holomorphic functions in the unit ball. Graduate Texts in Mathematics, 226. Springer-Verlag, New York, (2005).

[34] A. Zygmund: Trigonometric Series. 2nd ed. reprinted, I, II. Cambridge University Press (1979).