

MIXING OPERATORS WITH PRESCRIBED UNIMODULAR EIGENVALUES

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ABSTRACT. For arbitrary closed countable subsets Z of the unit circle examples of topologically mixing operators on Hilbert spaces are given which have a densely spanning set of eigenvectors with unimodular eigenvalues restricted to Z . In particular, these operators cannot be ergodic in the Gaussian sense.

1. INTRODUCTION AND MAIN RESULT

The dynamical behaviour of linear operators acting on Fréchet spaces X has been investigated intensively in the last years. Recommended introductions are the textbooks [3] and [9], and also the recent article [12]. It turns out that the richness of eigenvectors corresponding to unimodular eigenvalues strongly influences the metric dynamical properties of linear operators. In particular, a linear operator on a Hilbert space admits a Gaussian invariant measure of full support if and only if it has spanning unimodular eigenvectors and is ergodic in the Gaussian sense (that is, ergodic with respect to a Gaussian measure of full support) if and only if it has perfectly spanning unimodular eigenvectors. For these and corresponding results we refer in particular to [1], [2], [4], [10] and again to [12].

Due to recent deep results of Menet ([14]) and Grivaux, Matheron and Menet ([12]), in the situation of Hilbert spaces X the picture has become quite complete for the case of chaotic operators, that is, for hypercyclic operators having eigenvectors corresponding to roots of unity (i.e. periodic vectors) which span a dense subspace of X . Less is known, however, in the case of absence of periodic or almost periodic vectors (cf. [12, Section 1.3]). In [11, Question 3] (see also [12, Question 7.7]) it is asked if a hypercyclic operator with a densely spanning set of eigenvectors corresponding to rationally independent eigenvalues is already ergodic.

In this paper, we give examples of topologically mixing operators on Hilbert spaces which have a densely spanning set of eigenvectors with unimodular eigenvalues restricted to an arbitrary prescribed closed, countable subset Z of the unit circle \mathbb{T} . In particular, such operators cannot be ergodic in the Gaussian sense. Choosing Z as a rationally independent set, Question 3 from [11] can be answered in the negative, at least in the weak form that ergodicity in the Gaussian sense does not always follow from the existence of a spanning set of eigenvectors corresponding to a rationally independent set of unimodular eigenvalues.

For an open set Ω in the extended plane \mathbb{C}_∞ we denote by $H(\Omega)$ the Fréchet space of functions holomorphic in Ω and vanishing at ∞ endowed with the topology of locally uniform convergence, where, as usual, via stereographic projection we

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identify \mathbb{C}_∞ and the sphere \mathbb{S}^2 endowed with the spherical metric. We consider Bergman spaces on general open sets $\Omega \subset \mathbb{C}_\infty$: For $0 \leq p < \infty$ let $A^p(\Omega)$ be the space of all functions f holomorphic in Ω that satisfy

$$\|f\|_p := \|f\|_{\Omega,p} := \left(\int_{\Omega} |f|^p dm_2 \right)^{1/p} < \infty,$$

where m_2 denotes the spherical measure on \mathbb{C}_∞ . Then $A^0(\Omega) = H(\Omega)$ and for $p \geq 1$ the spaces $(A^p(\Omega), \|\cdot\|_p)$ are Banach spaces. In the case $p = 2$, the norm is induced by the inner product $(f, g) \mapsto \int_{\Omega} f \bar{g} dm_2$.

In the sequel we always consider open sets Ω with $0 \in \Omega$ in which case define $T = T_{A^p(\Omega)} : A^p(\Omega) \rightarrow A^p(\Omega)$ by

$$Tf(z) := (f(z) - f(0))/z \quad (z \neq 0), \quad Tf(0) := f'(0).$$

If $f(z) = \sum_{\nu=0}^{\infty} a_{\nu} z^{\nu}$, then

$$Tf(z) = \sum_{\nu=0}^{\infty} a_{\nu+1} z^{\nu}$$

for $|z|$ sufficiently small. We call T the Taylor (backward) shift on $A^p(\Omega)$. Writing $S_n f(z) := \sum_{\nu=0}^n a_{\nu} z^{\nu}$ for the n -th partial sum of the Taylor expansion $\sum_{\nu=0}^{\infty} a_{\nu} z^{\nu}$ of f about 0, by induction it is easily seen that, for $n \in \mathbb{N}_0$,

$$(1) \quad T^{n+1}f(z) = (f - S_n f(z))/z^{n+1} \quad (z \neq 0) \quad \text{and} \quad T^{n+1}f(0) = a_{n+1}.$$

In [5] it is shown that for open sets Ω with $0 \in \Omega$ the Taylor shift T is topologically mixing on $H(\Omega)$ if and only if each connected component of $\mathbb{C}_\infty \setminus \Omega$ meets \mathbb{T} . Results concerning topological and metric dynamics of the Taylor shift on Bergman spaces are proved in [6] and [17].

We write $M^* := (\mathbb{C}_\infty \setminus M)^{-1}$ for $M \subset \mathbb{C}_\infty$. Then Ω^* is a compact plane set (note that $0 \in \Omega$) and the spectrum of T is contained in Ω^* . In the case $1 \leq p < 2$ it is easily seen that $f \in A^p(\Omega)$ is an eigenfunction for T if and only if, for some $\alpha \in \Omega^*$, the function f is a scalar multiple of γ_α , where γ_α is defined by

$$\gamma_\alpha(z) = 1/(1 - \alpha z)$$

for $z \in \Omega$ (with $0 \cdot \infty := 0$). In this case, α is the corresponding eigenvalue and the spectrum as well as the point spectrum both equal Ω^* . If $p \geq 2$, then the functions γ_α still belong to $A^p(\Omega)$ for all α in the interior of Ω^* , but in general not for α belonging to the boundary of Ω^* . If Ω has small spherical measure near a boundary point $1/\alpha$ of Ω , it may, however, happen that γ_α is again an eigenfunction of T (that is, γ_α belongs to $A^p(\Omega)$). For example, $\gamma_1 \in A^2(\Omega)$ in the case of the crescent-shaped region $\Omega = \mathbb{D} \setminus \{z : |z - 1/2| \leq 1/2\}$ where \mathbb{D} denotes the open unit disc in \mathbb{C} . This opens up the possibility to place eigenvalues at certain points of the boundary of Ω^* . A corresponding construction leads to our main result. We write $\mathcal{E}(T)$ for the set of unimodular eigenvalues of T , which for $T = T_{A^2(\Omega)}$ equals the set of $\lambda \in \mathbb{T}$ such that $\gamma_\lambda \in A^2(\Omega)$.

Theorem 1.1. *Let $Z \subset \mathbb{T}$ be an infinite closed set. Then there is a open set $\Omega \subset \mathbb{D}$ so that the Taylor shift $T = T_{A^2(\Omega)}$ is topologically mixing, $\mathcal{E}(T) \subset Z$ and $\{\gamma_\lambda : \lambda \in \mathcal{E}(T)\}$ spans a dense subspace of $A^2(\Omega)$.*

Remark 1.2. Since Hilbert spaces are of cotype 2, the main theorem from [4] implies that, in the situation of Theorem 1.1, for countable Z the Taylor shift T is not ergodic in the Gaussian sense. So, as already mentioned above, Question 3

from [11] can be answered in the negative in so far as the existence of a spanning set of eigenvectors corresponding to a rationally independent set of unimodular eigenvalues does not always imply ergodicity in the Gaussian sense. We are left with the open question whether T is ergodic with respect to some measure of full support or (upper) frequently hypercyclic.

If in the situation of Theorem 1.1 the set Z consists of roots of unity, then the Taylor shift is chaotic with unimodular eigenvalues only in Z and not ergodic in the Gaussian sense. So we have an alternative construction for a chaotic operator on a Hilbert space that is not ergodic in the Gaussian sense. The first construction of such an operator on $\ell_2(\mathbb{N})$ given by Menet ([14]) even leads to an operator which is not (upper) frequently hypercyclic. The approach via Taylor shift is, however, quite different and gives more flexibility in prescribing unimodular eigenvectors.

2. PROOF OF THEOREM 1.1

As main tool for the proof of Theorem 1.1 we seek results on rational approximation in the mean. In the sequel, we restrict to open sets Ω which are bounded in \mathbb{C} or contain the point ∞ .

Remark 2.1. Let Ω be an open and bounded set in \mathbb{C} and suppose that each point on the boundary of Ω^* belongs to the boundary of some component of the interior $(\Omega^*)^\circ$ of Ω^* . Then Theorem 4 from [13] implies that the span of $\{\gamma_\alpha : \alpha \in (\Omega^*)^\circ\}$ is dense in $A^2(\Omega)$. If Ω is open in the extended plane with $\infty \in \Omega$, this is also the case (see [17, Remark 2.6]).

As indicated in the introduction, the function γ_α belongs to $A^2(\Omega)$ if α belongs to the boundary of Ω^* and Ω has small spherical measure near $1/\alpha$. For $k \in \mathbb{N}_0$, $z \in \Omega$ and $\alpha \in \mathbb{C}$ we write

$$\gamma_{\alpha,k}(z) := z^k / (1 - \alpha z)^{k+1} = z^k \gamma_\alpha^{k+1}(z).$$

We show that in the case of sufficiently small measure near $1/\alpha$ all functions $\gamma_{\alpha,k}$ belong to $A^2(\Omega)$ and that under appropriate conditions they span a dense subspace of $A^2(\Omega)$. The approach is strongly influenced by the proof of a result on the completeness of polynomials in $A^2(\Omega)$ (see [15, Theorem 12.1], cf. also [8, Chapter I, Section 3]).

Let $\delta > 0$ and let the "cup" C_δ be defined as the interior of the convex hull of $\{t + is(t) : -\delta \leq t \leq \delta\}$, where

$$s(t) := \exp(-\exp(1/|t|))$$

(with $s(0) = 0$). With that, we say that two components A, B of an open set $U \subset S := \mathbb{R} + i(-\pi/2, \pi/2)$ are directly bridged at $w \in S$, if $\omega \in \mathbb{T}$ and $\delta > 0$ exist with $w + \omega C_\delta \subset A$ and $w - \omega C_\delta \subset B$. If $\varphi : S \rightarrow \mathbb{S}^2$ is the standard parametrisation of $\mathbb{S}^2 \setminus \{\pm(0, 0, 1)\}$, that is,

$$\varphi(t + is) = (\cos(s) \cos(t), \cos(s) \sin(t), \sin(s))$$

for $t \in \mathbb{R}$ and $-\pi/2 < s < \pi/2$, we say that two components C, D of an open set $V \subset \mathbb{C}_\infty$ with $0, \infty \notin \partial V$ are directly bridged, if the corresponding inverse images under φ in S are bridged at some w . In this case $\zeta = \varphi(w)$ is said to be a bridge point for (C, D) . We say that C, D are bridged if finitely many components C_0, C_1, \dots, C_m exist with $C_0 = C$, $C_m = D$ and so that C_j, C_{j-1} are directly bridged. If \mathcal{C} denotes the set of components of V , then bridging induces an equivalence relation \sim on \mathcal{C} .

If a system $\mathcal{D} \subset \mathcal{C}$ is a complete system of representatives for \sim , we briefly say that the system is complete for V .

Theorem 2.2. *Let Ω be an open set in \mathbb{C}_∞ which is bounded in \mathbb{C} or contains ∞ and suppose that each point on the boundary of Ω^* belongs to the boundary of some component of the interior $(\Omega^*)^\circ$ of Ω^* . Moreover, suppose \mathcal{D} to be complete for $(\Omega^*)^\circ$ and for $D \in \mathcal{D}$ let α_D be either in D or a bridge point of (C, D) for some $C \in \mathcal{C}$. Then both $\{\gamma_\alpha : \alpha \in \bigcup_{D \in \mathcal{D}} D\}$ and $\{\gamma_{\alpha_D, k} : D \in \mathcal{D}, k \in \mathbb{N}_0\}$ have dense span in $A^2(\Omega)$.*

Proof. 1. Let $g \in A^2(\Omega) = A^2(\Omega)'$. Then the Cauchy transform $Vg : (\Omega^*)^\circ \rightarrow \mathbb{C}$ of g , defined by

$$(Vg)(\alpha) = \int_{\Omega} \gamma_\alpha(z) \bar{g}(z) dm_2(z)$$

for $\alpha \in (\Omega^*)^\circ$, is holomorphic with

$$(Vg)^{(k)}(\alpha) = k! \int_{\Omega} \gamma_{\alpha, k}(z) \bar{g}(z) dm_2(z)$$

for all $\alpha \in (\Omega^*)^\circ$ and $k \in \mathbb{N}_0$. According to the Hahn-Banach theorem and Remark 2.1, it suffices to show that $Vg|_C = 0$ for all $C \in \mathcal{C}$ under each of the two conditions stated in the theorem.

The crucial point is that Vg extends at bridge points ζ to a function which belongs to a quasi-analytic subclass of $C^\infty(I)$ for some line I with $I \setminus \{\zeta\} \subset (\Omega^*)^\circ$: Let ζ be a bridge point for (C, D) . In order to reduce notational complexity we assume that $\zeta = 1$ and that C lies in \mathbb{D} and D outside \mathbb{D} . Considering the fact that 1 is a bridge point, we may fix $0 < r < 1$ in such a way that the corresponding "cup"-sets $\pm C_r$ satisfy $\varphi^{-1}(\Omega) \cap \pm C_r = \emptyset$. Let

$$I = \varphi(i[-r/2, r/2])$$

(note that I is a compact interval in \mathbb{R} with 1 in its interior). To show that $Vg|_{I \setminus \{1\}}$ extends at 1 to a C^∞ -function on I and that the extension (again denoted by Vg) belongs to a quasi-analytic subclass of $C^\infty(I)$, we estimate the derivatives of Vg on $I \setminus \{1\}$. By the Cauchy-Schwarz inequality we have

$$(2) \quad |(Vg)^{(k)}(x)| \leq k! \int_{\Omega} |\gamma_{x, k} \bar{g}| dm_2 \leq k! \|g\|_2 \left(\int_{\Omega} |\gamma_{x, k}|^2 dm_2 \right)^{1/2}$$

for $k \in \mathbb{N}_0$ and $1 \neq x \in I$. So it suffices to estimate the latter integrals. We define

$$W_A(k, x) := \int_A |\gamma_{x, k}|^2 dm_2 = \int_A \frac{|z|^{2k} dm_2(z)}{|1 - xz|^{2k+2}}$$

for $k \in \mathbb{N}_0$, $x \in I$ and measurable $A \subset \Omega$. With $Q := [-r, r] + i[-r, r]$, we have

$$\sup_{x \in I} W_{\Omega \setminus \varphi(Q)}(k, x) = \mathcal{O}(q_1^k)$$

for some positive q_1 . To estimate $W_{\varphi(Q) \cap \Omega}$, we observe that the shape of Ω in $\varphi(Q)$ allows that with some constant $c > 0$ we have $|1 - xz| \geq c|1 - z|$ for all $x \in I$ and all $z \in \Omega \cap \varphi(Q)$. Thus, for $x \in I$ we obtain (by substituting $u = e^{1/t}$ in the last

step) with a positive constant q_2

$$\begin{aligned} W_{\Omega \cap \varphi(Q)}(k, x) &\leq q_2^k W_{\Omega \cap \varphi(Q)}(k, 1) \leq c \int_{-r}^r \int_{-s(t)}^{s(t)} \frac{\cos s}{|t + is|^{2k+2}} ds dt \\ &\leq 4q_2^k \int_0^r \frac{s(t)}{t^{2k+2}} dt = 4q_2^k \int_{e^{1/r}}^\infty e^{-u} u^{-1} \log^{2k}(u) du. \end{aligned}$$

For k sufficiently large and $u \geq k^2$ we have $\log^{2k}(u) \leq e^{u/2}$. Hence, by splitting up the integral at k^2 , one can see that

$$\int_{e^{1/r}}^\infty e^{-u} u^{-1} \log^{2k}(u) du = \mathcal{O}(k^2 \log^{2k}(k^2)) = \mathcal{O}(5^k \log^{2k}(k)).$$

Putting together we find that

$$(3) \quad \sup_{x \in I} W_\Omega(k, x) = \mathcal{O}(q_3^k \log^{2k}(k))$$

for some positive constant q_3 . Since $|\gamma_{x,k}(z)| \leq |\gamma_{1,k}(z)|$ for $z \in \varphi(Q)$ and $x \in I$, and since there is a constant c_k with $|\gamma_{x,k}(z)| \leq c_k$ for $z \notin \varphi(Q)$ and $x \in I$, the m_2 -integrability of g on Ω and of $|\gamma_{1,k}\bar{g}|$ on $\varphi(Q) \cap \Omega$ imply, by differentiation of parameter integrals, that Vg extends to a function in $C^\infty(I)$ (which we again denote by Vg). Moreover, combining (2) and (3) we obtain

$$\sup_{x \in I} |(Vg)^{(k)}(x)| = \mathcal{O}(k! \sqrt{q_3}^{-k} \log^k(k)).$$

Hence, the Denjoy-Carleman theorem shows that Vg belongs to a quasi-analytic subclass of $C^\infty(I)$.

2. Suppose that $g \perp \gamma_\alpha$ for all $\alpha \in \bigcup_{D \in \mathcal{D}} D$, that is $(Vg)|_D = 0$ for all $D \in \mathcal{D}$. If $\mathcal{D} = \mathcal{C}$ then $Vg = 0$. If $C \in \mathcal{C} \setminus \mathcal{D}$ then C, D are bridged for some $D \in \mathcal{D}$. We can assume that C, D are directly bridged with bridge point $\zeta = 1$ as in part 1 above. From the assumption $(Vg)|_D = 0$ we have $(Vg)^{(k)}(1) = 0$ for all k , and then also $(Vg)|_C = 0$ by quasi-analyticity and the identity theorem for holomorphic functions.

3. Suppose that $g \perp \{\gamma_{\alpha_D, k} : D \in \mathcal{D}, k \in \mathbb{N}_0\}$, that is, $\int_\Omega \gamma_{\alpha_D, k} \bar{g} dm_2 = 0$ for all $D \in \mathcal{D}$ and $k \in \mathbb{N}_0$, and let $C \in \mathcal{C}$. If $C \in \mathcal{D}$ then $\alpha_C \in C$ or α_C is a bridge point of C, E for some $E \in \mathcal{C}$. In both cases, according to part 1 of the proof (with $\alpha_C = 1$ without loss of generality) we have $(Vg)|_C = 0$. If $C \notin \mathcal{D}$, then there is $D \in \mathcal{D}$ so that C, D are bridged. Again, we can assume that C, D are directly bridged with bridge point $\zeta = 1$ as above. If $\alpha_D \in D$, then $(Vg)|_D = 0$. Hence $(Vg)^{(k)}(1) = 0$ for all k and then $(Vg)|_C = 0$. If α_D is a bridge point of (D, E) for some $E \in \mathcal{C}$, then by the same argument as above we can now conclude $(Vg)|_D = 0$. As in the first case, $(Vg)|_C = 0$. \square

Remark 2.3. If, for some component $D \in \mathcal{C}$, the single set system $\{D\}$ is complete for $(\Omega^*)^\circ$ and if α is a point in D or a bridge point of (C, D) for some component C , then the span of $\{\gamma_{\alpha, k} : k \in \mathbb{N}_0\}$ is dense in $A^2(\Omega)$. In particular, in case $\alpha = 0$ we conclude that the polynomials form a dense set in $A^2(\Omega)$ (cf. Theorem 12.1 in [15], where actually a weaker condition on the sharpness of Ω near $1/\alpha$ is proved to be sufficient).

From the Godefroy-Shapiro criterion (see e.g. [9, Theorem 3.1]) and Theorem 2.2 we obtain

Corollary 2.4. *Let Ω be an open set in \mathbb{C}_∞ which is bounded in \mathbb{C} or contains ∞ and suppose that each point in the boundary of Ω^* belongs to the boundary of some component of $(\Omega^*)^\circ$. If complete systems \mathcal{D} and \mathcal{D}' for $(\Omega^*)^\circ$ exist with $D \subset \mathbb{D}$ for all $D \in \mathcal{D}$ and $D' \subset \mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ for all $D' \in \mathcal{D}'$, then $T_{A^2(\Omega)}$ is topologically mixing.*

Example 2.5. Let $\Omega = \mathbb{C}_\infty \setminus (\overline{U} \cup \overline{G}) = ((1/\overline{U}) \cup (1/\overline{G}))^*$, where $G, 1/U \subset \mathbb{D}$ are domains with $0 \notin \overline{G}$ and so that U, G (or, equivalently, $1/G, 1/U$) are bridged. In this case, $\mathcal{D} = \{1/U\}$ and $\mathcal{D}' = \{1/G\}$ are complete systems for $(\Omega^*)^\circ$. By Corollary 2.4, the Taylor shift $T_{A^2(\Omega)}$ is mixing. If $0 \in 1/U$ (e.g. for $1/U = \mathbb{D}$, in which case $\Omega = \mathbb{D} \setminus \overline{G}$), the polynomials are dense in $A^2(\Omega)$ and the Taylor shift on $A^2(\mathbb{D})$ is a quasi-factor of the Taylor shift on $A^2(\Omega)$.

Theorem 2.6. *For each infinite closed set $Z \subset \mathbb{T}$ a domain $G \subset \mathbb{D}$ bridged to $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ exists with $\overline{G} \cap \mathbb{T} \subset Z$ and so that the span of $\{\gamma_\zeta : \zeta \in \overline{G} \cap \mathbb{T}\}$ is dense in $A^2(\mathbb{D} \setminus \overline{G})$.*

Proof. Let

$$U_r := \{z \in \mathbb{C} : |z - 1| < r\} \cap \mathbb{D}.$$

In our construction we will cut out sets from \mathbb{D} , essentially of the form $\zeta\varphi(C_\delta)$, that are "flat" at points $\zeta \in \mathbb{T}$ in such a way that these ζ become bridge points to $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$. We have to control the integrals of γ_ζ over the remaining areas next to such a bridge point. To this end, throughout the proof we fix a $\delta > 0$ and a corresponding $r_0 > 0$ that is sufficiently small such that for all $\zeta \in \mathbb{T} \cap \overline{U_{r_0}}$ the set $U_{r_0} \setminus \zeta\varphi(C_\delta)$ consists of the (one or two) component(s) bounded only by $\{\varphi(t + is(t)) : t \in [-\delta, \delta]\}$ and $\overline{U_{r_0}} \cap \mathbb{T}$, i.e. for every z in this set we find $t' \in [-\delta, \delta] \setminus \{0\}$ and $\zeta' \in \overline{U_{r_0}} \cap \mathbb{T}$ such that $\varphi(t' + is(t'))$ and ζ' define a line through the origin and the line segment between those points contains z and does not intersect with $\varphi(C_\delta)$. The definition of C_δ is extended to $C_{\delta, \rho}$, with $\rho > 0$, defined as the interior of

$$\text{conv}\{t + i ds(t) : -\delta \leq t \leq \delta, \rho \leq d \leq 1\},$$

with conv denoting the convex hull. Together, this allows us to decrease for $\zeta \in \mathbb{T} \cap \overline{U_{r_0}}$ the area of $U_r \setminus \zeta\varphi(C_{\delta, \rho})$ to become arbitrarily close to zero by decreasing ρ .

We may suppose that Z has an accumulation at 1. In the initial step of our construction we set $Z_0 := \emptyset$ and $G_0 = \varphi(C_{\delta, 1})$. This implies $\gamma_{1, k} \in A^2(\mathbb{D} \setminus \overline{G_0})$ for all k . For a subsequent $n \in \mathbb{N}$, the iterative procedure goes as follows. We choose $0 < r_n < r_{n-1}$ such that

$$(4) \quad \int_{U_{r_n} \setminus \overline{G_{n-1}}} |\gamma_{1, j}|^2 dm_2 < 1/n$$

for $j = 1, \dots, n$. By a variant of Runge's theorem on approximation by rational functions with first order poles, as it can be found in [16], we find a finite set

$$Z_n \subset \overline{U_{r_n/2}} \cap \left(Z \setminus \left(\bigcup_{j=0}^{n-1} Z_j \cup \{1\} \right) \right)$$

and an $R_j = R_{n, j} \in \text{span}\{\gamma_\zeta : \zeta \in Z_n\}$ such that

$$(5) \quad \max_{z \in \mathbb{D} \setminus U_{r_n}} |\gamma_{1, j}(z) - R_j(z)| < 1/n$$

for $j = 1, \dots, n$. Then we set

$$G_n := \bigcup_{\zeta \in Z_n} \zeta \varphi(C_{\delta, \rho_n})$$

where $\rho_n > 0$ is chosen in a way that the area of $U_{r_n} \setminus \overline{G_n}$ is sufficiently small to yield

$$(6) \quad \int_{U_{r_n} \setminus \overline{G_n}} |R_j|^2 dm_2 < 1/n.$$

This implies $\gamma_\zeta \in A^2(\mathbb{D} \setminus \overline{G_n})$ for all $\zeta \in Z_n$ and (4), (5), (6) give

$$\int_{\mathbb{D} \setminus \overline{G_n}} |R_j(z) - \gamma_{1,j}(z)|^2 dm_2(z) < 3/n$$

for $j = 1, \dots, n$. For $G := \bigcup_{n \in \mathbb{N}} G_n$ we finally have $\gamma_\zeta \in A^2(\mathbb{D} \setminus \overline{G})$ for all $\zeta \in \mathbb{T} \cap \overline{G}$, and for every non negative integer k the function $\gamma_{1,k}$ belongs to the closure of the span of $\{\gamma_\zeta : \zeta \in \overline{G} \cap \mathbb{T}\}$. According to Example 2.5 with $1/U = \mathbb{D}$, the latter implies the denseness of $\text{span}\{\gamma_\zeta : \zeta \in \overline{G} \cap \mathbb{T}\}$ in $A^2(\mathbb{D} \setminus \overline{G})$. \square

Proof of Theorem 1.1. Let $\Omega = \mathbb{D} \setminus \overline{G}$ where G is as in Theorem 2.6. Then $\overline{G} \cap \mathbb{T} \subset \mathcal{E}(T)$. According to Example 2.5 and Theorem 2.6, T is topologically mixing. Since each point in $\mathbb{T} \setminus \overline{G}$ is an interior point of \mathbb{D} , no point in $\mathbb{T} \setminus \overline{G}$ belongs to the point spectrum, that is $\mathcal{E}(T) \subset Z$. \square

Remark 2.7. 1. By deleting sufficiently small parts from G it is possible to modify G to an open set W in such a way that $\Omega = \mathbb{D} \setminus \overline{W}$ is connected and the Taylor shift $T_{A^2(\Omega)}$ satisfies the same conditions as in Theorem 1.1.

2. The statement (and proof) of Theorem 2.6 can be modified in such a way that $\Omega^* \cap \mathbb{T} \subset Z$, i.e. the spectrum intersects \mathbb{T} only in Z (cf. [11, Question 3]).

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