Generic behaviour of classes of Taylor series outside the unit disc

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Abstract

It is known that, generically, Taylor series of functions holomorphic in the unit disc turn out to be "maximally divergent" outside of the disc. For functions in classical Banach spaces of holomorphic functions as for example Hardy spaces or the disc algebra the situation is more delicate. In this paper, it is shown that the behaviour of the partial sums on sets outside the open unit disc sensitively depends on the way the sets touch the unit circle. As main tools, results on simultaneous approximation by polynomials are proved.

Key words: Hardy spaces, disc algebra, universal functions

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1 Introduction

Let $\mathbb{C}_\infty$ be the extended complex plane. For an open set $\Omega \subset \mathbb{C}_\infty$ we denote by $H(\Omega)$ the Fréchet space of functions holomorphic in Ω (and vanishing at ∞ if $\infty \in \Omega$) endowed with the topology of locally uniform convergence. If $0 \in \Omega$ and $f \in H(\Omega)$, we write

$$s_n f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$$

with $a_n = a_n(f) = f^{(n)}(0)/n!$ for the n-th partial sum of the Taylor expansion $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ of f about 0. A classical question in complex analysis is how the partial sums $s_n f$ behave outside the disc of convergence and in particular on the boundary of the disc. Based on Baire's category theorem, it can be shown that for functions f holomorphic in the unit disc $\mathbb{D}$ generically the sequence $(s_n f)$ turns out to be universal outside of $\mathbb{D}$. For precise definitions and a large number of corresponding results, we refer in particular to the expository article [21]. For results on universal series in a more general framework see also [2].

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The situation changes if we consider classical Banach spaces of holomorphic functions. Our aim is to study the generic behaviour of the Taylor sections $s_n f$ outside of the open unit disc for functions in Hardy spaces and in the disc algebra.

2 Taylor series of functions in Hardy spaces

Let m denote the normalized arc length measure on the unit circle $\mathbb{T}$. For $1 \leq p < \infty$, the Hardy space $(H^p, ||\cdot||_p)$ is defined as the (Banach) space of all $f \in H(\mathbb{D})$ such that

$$||f||_p := \sup_{0<r<1} \Big(\int_{\mathbb{T}} |f_r|^p dm \Big)^{1/p} < \infty \ ,$$

where $f_r(z) := f(rz)$ for $z \in \mathbb{D}$. It is well known that each $f \in H^p$ has nontangential limits $f^*(z)$ at m-almost all z on the unit circle $\mathbb{T}$ and that $f^* \in L^p(\mathbb{T})$. Moreover, the mapping $f \mapsto f^*$ establishes an isometry between H^p and the closure of the polynomials in $L^p(\mathbb{T})$. As usual, we identify f and f^* and, in this way, H^p and the corresponding closed subspace of $L^p(\mathbb{T})$. In particular, the restrictions $s_n f|\mathbb{T}$ of the partial sums of the Taylor expansion of f are the partial sums of the Fourier expansion of f. For proofs of the above results and further properties we refer to [11] and [26].

According to the classical Carleson-Hunt theorem, for each $p > 1$ and each $f \in H^p$ the partial sums $s_n f$ converge to f almost everywhere on $\mathbb{T}$. Due to results of Kolmogorov, in the case $p = 1$ we have convergence in measure and therefore, in particular, each subsequence of $(s_n f)$ has a subsubsequence converging almost everywhere to f. Recently, Gardiner and Manolaki ([15]) have shown that, for general functions $f \in H(\mathbb{D})$, under the mere assumption of existence of a nontangential limit function f^* almost everywhere on a subarc of $\mathbb{T}$, no limit function differing from f^* on a set of positive measure on the corresponding arc can exist.

On the other hand, in [5] it is proved that for functions in H^p the partial sums generically turn out to have a maximal set of limit functions on closed sets of measure zero. In order to formulate the result more precisely, it is convenient to say that a property is satisfied for generically many elements of a complete metric space, if the property is satisfied on a residual set in the space. Then Theorem 1.1 from [5] states that, for each $p \geq 1$, each compact set $E \subset \mathbb{T}$ with vanishing arc length measure and each infinite subset Λ of $\mathbb{N}_0$, generically many f in H^p enjoy the property that for each continuous function $g : E \to \mathbb{C}$ a subsequence of $(s_n f)_{n\in\Lambda}$ tends to g uniformly on E.

Our aim is to consider, more generally, compact sets $E \subset \mathbb{C} \setminus \mathbb{D}$. As usual, for E compact in $\mathbb{C}$ we write

$$A(E) := \{h \in C(E) : h \text{ holomorphic in } E^0\}.$$

According to Mergelyan's Theorem, for E with connected complement, $A(E)$ is the closure of the set of polynomials in $C(E)$, where $C(E)$ is endowed with the uniform norm $||\cdot||_E$.

We denote $\mathbb{D}^* := \mathbb{C}_\infty \setminus \overline{\mathbb{D}}$. For compact sets $E \subset \mathbb{D}^*$ with connected complement, it is known that that generically many functions f in H^p – and even in the (smaller) disc algebra $A(\overline{\mathbb{D}})$ – have the property that the set of partial sums $\{s_n f|E : n \in \mathbb{N}\}$ is dense in $A(E)$ (see e.g. [21, Theorem 4.2]). In view of Theorem 1.1 from [5] mentioned above, a reasonable guess would be that this also holds for $E \subset \mathbb{C} \setminus \mathbb{D}$ with connected complement and touching $\mathbb{T}$ in a set of vanishing m-measure. It turns out that this in *not* true:

Theorem 2.1. *Let $1 \leq p < \infty$ and let $E \subset \mathbb{C} \setminus \mathbb{D}$ compact such that E contains a rectifiable arc $\gamma : [0,1] \to \mathbb{C} \setminus \mathbb{D}$ with $\gamma(\{0,1\}) \subset \mathbb{T}$ and $\gamma(0) \neq \gamma(1)$. Then there does not exist an $f \in H^p$ such that the set $\{s_n f|E : n \in \mathbb{N}\}$ is dense in $A(E)$.*

Proof. The proof follows the main idea of the proof of the claim in [1] on p. 241. We write $B_\delta(a)$ for the closed disk about a of radius δ.

Let $f \in H^p$ and let $F \in H(\mathbb{D})$ denote an antiderivative of f. From $f \in H^p \subset H^1$ it follows that F extends continuously to $\overline{\mathbb{D}}$ (see e.g. [11, Theorem 3.11]). Denoting by (σ_n) the Cesàro means of (s_n) and writing $w_j := \gamma(j)$ for $j = 0, 1$, we conclude from Fejér's theorem that $\sigma_n F(w_j)$ converges to $l_j := F(w_j)$, for $j = 0, 1$. Moreover, we have

$$\sigma_n F(z) = s_n F(z) - \frac{z}{n+1} s_{n-1} f(z) \quad (z \in \mathbb{C}) \tag{1}$$

(cf. [1, p. 241]), and due to $f \in H^p$ we obtain $s_n f(w_j) = O(n^{1/p})$, $j = 0, 1$ (see e.g. [3, p. 388]). Hence, there exists some $r > 0$ such that both sequences $(w_j \cdot s_{n-1} f(w_j)/(n+1))_{n \in \mathbb{N}}$ are bounded by r. We now fix a polynomial p with

$$\int_\gamma p(z)dz \notin B_{2r}(l_1 - l_0).$$

Assuming that the set $\{s_n f|E : n \in \mathbb{N}\}$ is dense in $A(E)$, there would exist a strictly increasing sequence (n_k) of positive integers such that

$$||s_{n_k - 1} f - p||_{\gamma([0,1])} \to 0.$$

Integrating along γ, we would obtain

$$s_{n_k} F(w_1) - s_{n_k} F(w_0) \to \int_\gamma p(z)dz.$$

Furthermore, it would follow from equation (1) and the subsequent considerations that there exist a $q_0 \in B_r(0)$ and a subsequence (m_k) of (n_k) with $s_{m_k} F(w_0) \to l_0 + q_0$. Analogously, we could find some $q_1 \in B_r(0)$ and a subsequence (t_k) of (m_k) with $s_{t_k} F(w_1) \to l_1 + q_1$. Hence, we would obtain

$$s_{t_k} F(w_1) - s_{t_k} F(w_0) \to l_1 + q_1 - (l_0 + q_0)$$

and thus

$$\int_\gamma p(z)dz = l_1 + q_1 - (l_0 + q_0) \in B_{2r}(l_1 - l_0),$$

a contradiction. $\square$

Remark 2.2. The proof of Theorem 2.1 simplifies in case of $p > 1$ because in this situation, it follows from $s_n f(w_j) = O(n^{1/p})$ that we have convergence $s_n F(w_j) \to l_j$ for $j = 0, 1$ (cf. equation (1)).

Theorem 2.1 shows that even for nice compact sets E touching $\mathbb{T}$ in only two points as e.g. $E := \{z : |z-1| = 1\} \setminus \mathbb{D}$, the subarc of $|z-1| = 1$ lying outside the unit disc, we do not have denseness of the partial sums $s_n f|E$ in $A(E) = C(E)$ for any $f \in H^p$. This raises the question which extra conditions on E guarantee universality of $(s_n f|E)$ for generic $f \in H^p$.

According to the Cantor-Bendixson theorem (see, e.g. [23, Theorem 6.4]), each compact set $A \subset \mathbb{C}$ can be decomposed in unique way as union of a perfect set, the so-called perfect kernel of A, and a countable set. For compact $E \subset \mathbb{C} \setminus \mathbb{D}$ and $\zeta \in E \cap \mathbb{T}$ we write $C_E(\zeta)$ for the component of E that contains ζ and P_E for the union of all $C_E(\zeta)$ with ζ ranging over the perfect kernel of $E \cap \mathbb{T}$. We say that E satisfies the *kernel condition* if P_E has vanishing area measure and if, in addition, the set of all $\zeta \in E \cap \mathbb{T}$ with the property that $C_E(\zeta)$ has positive area measure has positive distance to the perfect kernel of $E \cap \mathbb{T}$.

With that we can state our main result:

Theorem 2.3. *Let $1 \le p < \infty$ and E a compact subset of $\mathbb{C} \setminus \mathbb{D}$ that has connected complement in $\mathbb{C}$ and satisfies the kernel condition. Moreover, suppose that no component of E touches $\mathbb{T}$ in more than one point and that $E \cap \mathbb{T}$ has vanishing arc length measure. Then for all infinite sets $\Lambda \subset \mathbb{N}_0$ generically many f in H^p enjoy the property that for each $g \in A(E)$ a subsequence of $(s_n f)_{n \in \Lambda}$ tends to g uniformly on E.*

We give examples illustrating the conditions imposed on E in Theorem 2.3. According to the above considerations, the condition that $E \cap \mathbb{T}$ has vanishing arc length measure turns out to be necessary and the condition that no component of E touches $\mathbb{T}$ in more than one point is at least a natural one. We do not know if the kernel condition is necessary in any sense. Note, however, that this condition is not very restrictive. It is satisfied in particular if the perfect kernel of $E \cap \mathbb{T}$ is empty, which means that $E \cap \mathbb{T}$ is countable, and if P_E has vanishing area measure and no component $C_E(\zeta)$ has positive area measure.

Examples 2.4. 1. Let C be the classical Cantor $(1/3)$-set in $[0, 1]$ and let C_n be the n-th iterate of the Cantor set, that is,

$$C_n := \bigcup_{k=0}^{3^{n-1}-1} \left(\left[\frac{3k}{3^n}, \frac{3k+1}{3^n} \right] \cup \left[\frac{3k+2}{3^n}, \frac{3k+3}{3^n} \right] \right).$$

For the set

$$E := e^{2\pi i C} \cup \bigcup_{n\in\mathbb{N}} (1+1/n)e^{2\pi i C_n}$$

we have $P_E = e^{2\pi i C}$ and E satisfies the conditions of Theorem 2.3. By "blowing up" the arcs of $(1+1/n)e^{2\pi i C_n}$ to small sectors, E can be modified in such a way that each component of E lying in $\mathbb{D}^*$ has nonempty interior.

2. ("Cantor sun") For the compact set

$$E := [1,2]e^{2\pi i C}$$

we have $P_E = E$ and E satisfies the conditions of Theorem 2.3. The same remains true if one adds further disjoint compact parts to E that lie in $\mathbb{D}^*$ and have connected complement.

3. If E consists of countably many pairwise disjoint closed discs, each of which touches $\mathbb{T}$ in one point, then $P_E = \emptyset$ and E satisfies the conditions of Theorem 2.3. Again, the same remains true if one adds further disjoint compact parts to E that lie in $\mathbb{D}^*$ and have connected complement.

4. If E consists of the Cantor sun and countably many pairwise disjoint closed discs, each of which touches $\mathbb{T}$ in one point, and all lying in a sector $|\arg(-z)| \le \alpha$, for some $\alpha < \pi/3$, then E still satisfies the assumptions of Theorem 2.3. This is no longer true for $\alpha = \pi/3$.

The proof of Theorem 2.3 is based on results on simultaneous approximation by polynomials. For the case of sets E in $\mathbb{T}$, this goes back to Havin ([18], see also [20]). We consider Banach spaces $X = (X, ||\cdot||_X)$ with $X \subset L^1(\sigma)$ for some Borel set $M = M_X \subset \mathbb{C}$ and some finite Borel measure $\sigma = \sigma_X$ supported on M. Moreover, we always suppose that the polynomials are dense in X and that

$$r_X := \limsup_{n\to\infty} ||P_n||_X^{1/n} < \infty$$

with $P_n(z) := z^n$. In the sequel, we call such spaces analytic. In particular, analytic spaces are separable since the polynomials with (Gaußian) rational coefficients also form a dense subset. The space H^p is analytic (with $\sigma = m$ the normalized arc length measure). If $E \subset \mathbb{C}$ is compact with connected complement then, according to Mergelyans's theorem, also the closed subspace $A(E)$ of $(C(E), ||\cdot||_E)$ is analytic.

We write $X_1 \oplus X_2$ for the (external) direct sum of two Banach spaces X_1 and X_2.

Lemma 2.5. *Let X and Y be analytic, with $X \subset H(\mathbb{D})$ and so that $f \mapsto a_n(f)P_n|M_Y$ is a continuous linear mapping from X to Y for all n. If the pairs $(P,P) := (P|M_X, P|M_Y)$, where P ranges over the set of polynomials, form a dense set in the sum $X \oplus Y$, then for all infinite sets $\Lambda \subset \mathbb{N}_0$ generically many f in X enjoy the property that for each $g \in Y$ a subsequence of $(s_n f)_{n\in\Lambda}$ tends to g in Y.*

Proof. We consider the family $(s_n)_{n\in\Lambda}$ (more precisely $f \mapsto s_n f|M_Y$) of continuous linear mappings from X to Y. As mentioned above, Y is separable. The Universality Criterion (see e.g. [16, Theorem 1] or [17, Theorem 1.57]) implies that it is sufficient – and necessary – to show that for each pair $(f,g) \in X \oplus Y$ and each $\varepsilon > 0$ there exist a polynomial P and an integer $n \in \Lambda$ so that $||f - P||_X < \varepsilon$ and $||g - s_n P||_Y < \varepsilon$. Since $s_n P = P$ for all polynomials P and all large n (depending on the degree of P), it suffices to show that the pairs of the form (P,P), where P ranges over the set of polynomials, form a dense set in $X \oplus Y$. □

In view of Lemma 2.5, Theorem 2.3 is an immediate consequence of the following result on simultaneous polynomial approximation.

Theorem 2.6. *Let $1 \le p < \infty$ and E a compact subset of $\mathbb{C} \setminus \mathbb{D}$ that has connected complement in $\mathbb{C}$ and satisfies the kernel condition. Moreover, suppose that no component of E touches $\mathbb{T}$ in more than one point and that $E \cap \mathbb{T}$ has vanishing arc length measure. Then the pairs (P,P), where P ranges over the set of polynomials, form a dense set in $H^p \oplus A(E)$.*

The remaining part of the section is devoted to the proof of Theorem 2.6.

Let X be analytic. By X' we denote the (norm) dual of X and by $H(\infty)$ the linear space of germs of functions holomorphic (and vanishing) at ∞. Then the Cauchy transform $K_X : X' \to H(\infty)$ with respect to X is defined by

$$(K_X\phi)(w) = \sum_{\nu=0}^{\infty} \phi(P_\nu)/w^{\nu+1} \quad (|w| > r_X,\ \phi \in X').$$

Since the polynomials form a dense set in X, the Hahn-Banach theorem implies that K_X is injective. Then the range $R(K_X)$ of K_X is the so-called Cauchy dual of X. The following consequence of the Hahn-Banach theorem (cf. [20, Theorem 1.2]) is the basis of our subsequent considerations.

Lemma 2.7. *Let X and Y be analytic. Then $R(K_X) \cap R(K_Y) = \{0\}$ if and only if the pairs (P,P), where P ranges over the set of polynomials, form a dense set in the sum $X \oplus Y$.*

Proof. Consider a functional $(\phi,\psi) \in (X \oplus Y)' = X' \oplus Y'$. Then we have

$$0 = (\phi,-\psi)(P_n,P_n) = \phi(P_n) - \psi(P_n)$$

for all $n \in \mathbb{N}_0$ if and only if $K_X\phi = K_Y\psi$.

If $R(K_X) \cap R(K_Y) = \{0\}$ then $K_X\phi = K_Y\psi = 0$. Since K_X and K_Y are injective, we obtain that $(\phi,\psi) = (0,0)$. Then the denseness of the span of the (P_n,P_n) follows from the Hahn-Banach theorem.

If, conversely, the span of (P_n,P_n) is dense in $X \oplus Y$ and if ϕ and ψ are so that $K_X\phi = K_Y\psi$, then the Hahn-Banach theorem implies that $(\phi,-\psi) = (0,0)$ and thus also $K_X\phi = K_Y\psi = 0$. □

In order to apply the lemma we need more information about the Cauchy transforms involved.

Let $E \subset \mathbb{C}$ be compact with connected complement. The Riesz representation theorem says that $C(E)'$ is isometrically isomorphic to the space of complex Borel measures supported on E and endowed with the total variation norm. If we identify $\psi \in C(E)'$ with the corresponding Borel measure μ, then $\mu(P_n) = \int_E \zeta^n d\mu(\zeta)$, for all $n \in \mathbb{N}_0$. Since $A(E)$ is a subspace of $C(E)$, the Hahn-Banach theorem implies that each $\psi \in A(E)'$ can also be represented by some μ (not uniquely, however). If μ is an arbitrary representing measure of ψ, the Cauchy transform of ψ is the germ given by

$$(K\psi)(w) := (K_{A(E)}\psi)(w) = \int_E \frac{d\mu(z)}{w - z} \quad (w \in \mathbb{C}_\infty \setminus E).$$

Similarly, according to $\phi(f) = \int_{\mathbb{T}} f\overline{h}dm$ ($f \in L^p(\mathbb{T})$) we may identify $\phi \in (L^p(\mathbb{T}))'$ with a unique function $h \in L^q(\mathbb{T})$, where q is the conjugate exponent of p (i.e. $pq = p + q$ for $p > 1$ and $q = \infty$ for $p = 1$). Again, in this way each $\phi \in (H^p)'$ has representations of the form $\overline{h}m$. From this it is seen that for $\phi \in (H^p)'$

$$(K\phi)(w) := (K_{H^p}\phi)(w) = \int_{\mathbb{T}} \frac{\overline{h}(z)}{w - z}\, dm(z) \quad (w \in \mathbb{D}^*),$$

where $\overline{h}m$ is any representing measure. It is known (cf. [8] or [11]) that for $p > 1$ the Cauchy dual of H^p is given by $H_0^q(\mathbb{D}^*)$ in the sense that each germ $K\phi$ in $R(K_{H^p})$ is given by a function $\Phi \in H_0^q(\mathbb{D}^*)$, where

$$H_0^q(\mathbb{D}^*) := \{g \in H(\mathbb{D}^*) : w \mapsto w^{-1}g(w^{-1}) \in H^q\}.$$

More precisely, if ϕ is represented by $\overline{h}m$, it turns out that

$$\Phi(w) = w^{-1}(Ph)(w^{-1}),$$

where $P : L^q(\mathbb{T}) \to H^q$ denotes the Riesz projection operator. In a similar way, for $p = 1$, the Cauchy dual is given by the space $BMOA$.

For $E \subset \mathbb{C} \setminus \mathbb{D}$ compact with connected complement in $\mathbb{C}$, let U_E denote the Runge hull of the component G of $\mathbb{D}^* \setminus E$ containing ∞, that is, the union of G and the compact components of $\mathbb{D}^* \setminus G$. Then U_E is a simply connected domain in $\mathbb{C}_\infty$ containing ∞.

Lemma 2.8. *Let $E \subset \mathbb{C} \setminus \mathbb{D}$ be compact with connected complement in $\mathbb{C}$. If $\psi \in A(E)'$ is so that there exists a function $\Phi \in H(\mathbb{D}^*)$ satisfying $\Phi = K\psi$ near ∞, then for each representing measure μ of ψ we have*

$$\psi(P_n) = \int_E z^n d\mu(z) = \int_{E \setminus U_E} z^n d\mu(z) \quad (n \in \mathbb{N}_0),$$

that is, $1_{E \setminus U_E}\mu$ is a representing measure for ψ with support in $E \setminus U_E$.

Proof. As mentioned above, U_E is a simply connected domain in $\mathbb{C}_\infty$ containing ∞. Let (γ_j) be a sequence of piecewise smooth Jordan curves in $\mathbb{D}^* \setminus E$ so that the interior domains $\mathrm{int}(\gamma_j)$ of γ_j decrease to $\mathbb{C}_\infty \setminus U_E$. The existence of such a sequence (γ_j) follows from the existence of a nested exhaustion of $\mathbb{D}^* \setminus E$ by domains G_j bounded by finitely many piecewise smooth Jordan curves. Then γ_j may be taken to be the part of the boundary of G_j containing $\mathbb{C}_\infty \setminus U_E$ in its interior.

We set $E_j := E \cap \mathrm{int}(\gamma_j)$. Supposing the curves γ_j to be positively oriented, Fubini's and Cauchy's theorems show that, for $n \in \mathbb{N}_0$, $j \in \mathbb{N}$, and R sufficiently large,

$$\begin{aligned}
2\pi i \int_E z^n d\mu(z) &= \int_E \int_{|\zeta|=R} \frac{\zeta^n}{\zeta - z}\, d\zeta\, d\mu(z) = \int_{|\zeta|=R} \zeta^n K\psi(\zeta) d\zeta \\
&= \int_{|\zeta|=R} \zeta^n \Phi(\zeta) d\zeta = \int_{\gamma_j} \zeta^n \Phi(\zeta) d\zeta = \int_{\gamma_j} \zeta^n K\psi(\zeta) d\zeta \\
&= \int_E \int_{\gamma_j} \frac{\zeta^n}{\zeta - z}\, d\zeta\, d\mu(z) = 2\pi i \int_{E_j} z^n d\mu(z)\ .
\end{aligned}$$

Since E_j decreases to $E \setminus U_E$, Lebesgue's dominated convergence theorem shows that

$$\psi(P_n) = \int_E z^n d\mu(z) = \int_{E \setminus U_E} z^n d\mu(z) \qquad (n \in \mathbb{N}_0).$$

$\square$

Lemma 2.9. *Let E be a compact subset of $\mathbb{C} \setminus \mathbb{D}$ with connected complement in $\mathbb{C}$. If X is an analytic space with $r_X \le 1$, then the set $\{(P, P) : P$ polynomial$\}$ is dense in $X \times A(E)$ if (and only if) it is dense in $X \times A(E \setminus U_E)$.*

Proof. Let $\phi \in X'$ and $\psi \in A(E)'$ with representing measure μ and so that $K\phi = K\psi$. Since $r_X \le 1$, there is a function $\Phi \in H(\mathbb{D}^*)$ with $\Phi = K\phi (= K\psi)$ near ∞. According to Lemma 2.8, the Borel measure $\nu := 1_{E \setminus U_E} \mu$ is a representing measure for ψ having support in $E \setminus U_E$. Hence, ν defines a continuous linear functional on $A(E \setminus U_E)$ with Cauchy transform $K\psi$. From the assumption and Lemma 2.7 it follows that $K\psi = K\phi = 0$ and then a further application of Lemma 2.7 implies the statement. $\square$

Proof of Theorem 2.6. Since H^p is densely and continuously embedded into H^1, for all $p > 1$, we can restrict to the case $p > 1$. As usual, for closed $A \subset \mathbb{T}$ and $A^{(0)} := A$ we write $A^{(n+1)}$ for the Cantor-Bendixson derivative of $A^{(n)}$, that is, the set of limit points of $A^{(n)}$.

Due to Lemmas 2.5, 2.7, and 2.9, it suffices to show that for $E_1 := E \setminus U_E$

$$R(K_{H^p}) \cap R(K_{A(E_1)}) = \{0\}.$$

To this aim, we consider $\phi \in (H^p)'$ and $\psi \in A(E_1)'$ represented by $\overline{h}m$ and μ, respectively, and satisfying $K\phi = K\psi$ in $H(\infty)$. Then $K\phi$ is the germ of a function $\Phi \in H(\mathbb{D}^*)$ and $K\psi$ is the germ of a function $\Psi \in H(\mathbb{C}_\infty \setminus E_1)$. Since no component of E_1 meets $\mathbb{T}$ in more than one point, there is a common holomorphic extension to $\mathbb{C}_\infty \setminus (E \cap \mathbb{T})$, which we also denote by Φ.

By Q_E we denote the closure of the union of all components $C_E(\zeta)$ that have positive area measure. As a consequence of the Sura-Bura Theorem and the second requirement of the kernel condition, Q_E and P_E have positive distance. Since P_E has vanishing area measure, by a well known result of Carleson (see, e.g. [14, Chapter II, Theorem 8.2]), the support of μ is restricted to $(E\cap\mathbb{T})\cup Q_E$.

Moreover, the Cauchy integral $K\psi$ defines a locally integrable function with respect to plane area measure (see e.g. [10, pp. 192]). Hence, $\Phi|\mathbb{D} = K\psi|\mathbb{D}$ is integrable with respect to area measure. Since functions in $H_0^q(\mathbb{D}^*)$ are in particular locally integrable with respect to area measure, this shows that Φ is locally integrable with respect to plane area measure. But then, by estimating the Laurent coefficients, it can be shown that Φ can have only poles of first order or removable singularities at isolated points of $E \cap \mathbb{T}$ (cf. [9, p. 112, Exercise 17]). The fact that $\Phi \in H_0^q(\mathbb{D}^*)$ rules out the case of poles, so that Φ extends to a function holomorphic in $\mathbb{C}_\infty \setminus (E \cap \mathbb{T})^{(1)}$. By induction, Φ extends to a function holomorphic in $\mathbb{C}_\infty \setminus (E \cap \mathbb{T})^{(n)}$, for all $n \in \mathbb{N}$. Since $E \cap \mathbb{T}$ is compact, we have $\bigcap_{n\in\mathbb{N}}(E \cap \mathbb{T})^{(n)} = F$, where F denotes the perfect kernel of $E \cap \mathbb{T}$ (see e.g. [23, Theorem (6.11) and the subsequent considerations]). By a similar (but simpler) argument as in the proof of Lemma 2.8 this implies that $1_F\mu$ is also a representing measure for ψ.

Since $K\phi = K\psi$, the measure $\nu := 1_F\mu - \overline{h}m$ with support in $\mathbb{T}$ satisfies $\nu(P_n) = 0$ for all $n \in \mathbb{N}_0$. The F. and M. Riesz theorem then implies that ν is absolutely continuous with respect to m and therefore the same is true for $1_F\mu$. On the other hand, since $m(F) = m(E \cap \mathbb{T}) = 0$, the measure $1_F\mu$ is singular with respect to m. This shows that $1_F\mu = 0$ and from that we obtain $K\phi = K\psi = 0$. □

3 Taylor series of functions in the disc algebra

We consider the disc algebra $A(\overline{\mathbb{D}}) = \{f \in C(\overline{\mathbb{D}}) : f|\mathbb{D} \text{ holomorphic}\}$. As already indicated above, it is known that generically many functions in the disc algebra satisfy the property that $\{s_nf|E : n \in \mathbb{N}_0\}$ is dense in $A(E)$ for all compact sets $E \subset \mathbb{D}^*$ with connected complement (see e.g. [21, Theorem 4.2]).

According to Fejér's theorem, the Taylor series of each function $f \in A(\overline{\mathbb{D}})$ is Cesàro summable to f on $\mathbb{T}$, that is, the Cesàro means σ_nf of the s_nf converge to f on $\mathbb{T}$ (even uniformly). As a consequence, for functions $f \in A(\overline{\mathbb{D}})$ the limit behaviour of the sequence of partial sums (s_nf) on sets $E \subset \mathbb{C} \setminus \mathbb{D}$ touching $\mathbb{T}$ is significantly more restricted than in the case of functions in the Hardy spaces. In the first result we will show now that even pointwise universality does not appear on sets E touching $\mathbb{T}$ in a single point and not being negligible in the sense of logarithmic capacity outside the closed unit disc.

Let $f \in H(\mathbb{D})$ with $f(z) = \sum_{\nu=0}^\infty a_\nu z^\nu$ for $z \in \mathbb{D}$. If (n_k) is a strictly increasing sequence in $\mathbb{N}$ we say that f has Hadamard-Ostrowski gaps relative to (n_k) if a sequence of integers (p_k) with the following properties exists:

- $n_{k-1} \le p_k < n_k$ and $\limsup_{k\to\infty} p_k/n_k < 1$,
- If J is the set of integers ν with $p_k < \nu \le n_k$ for some k, then

$$\limsup_{J \ni \nu \to \infty} |a_\nu|^{1/\nu} < 1.$$

With this notation we have

Lemma 3.1. *Let $f \in H(\mathbb{D})$ be such that the sequence $((s_n f(\zeta))$ is Cesàro summable to s for some $\zeta \in \mathbb{T}$ and let (n_k) be a strictly increasing sequence in $\mathbb{N}$. If f has Hadamard-Ostrowski gaps relative to (n_k) and if $(s_{n_k} f(\zeta))_k$ converges then the limit equals s.*

Proof. Let (p_k) and J be as above. Then the power series

$$g(z) := \sum_{\nu \in J} a_\nu z^\nu$$

has radius of convergence > 1. If $f_0 := f - g$, then the Taylor coefficients of f_0 vanish for $p_k < \nu \le n_k$. This implies that $s_n f_0 = s_{n_k} f_0$ for all $p_k < n \le n_k$.

If c denotes the limit of $(s_{n_k} f(\zeta))_k$ then the sequence $(s_{n_k} f_0(\zeta))_k$ converges to $c_0 := c - g(\zeta)$. Moreover, the Cesàro means $\sigma_n f_0(\zeta)$ tend to $s_0 := s - g(\zeta)$, as $n \to \infty$. Passing to a subsequence of (n_k) we may suppose that (p_k/n_k) converges to some $\alpha < 1$. Then we have

$$\sigma_{n_k} f_0(\zeta) = \frac{p_k}{n_k} \sigma_{p_k} f_0(\zeta) + (1 - \frac{p_k}{n_k}) s_{n_k} f_0(\zeta)$$

and the left hand side converges to s_0 and the right hand side to $\alpha s_0 + (1-\alpha) c_0$, as $k \to \infty$. This implies $c_0 = s_0$ and thus $c = s$. □

As a consequence we obtain

Theorem 3.2. *Let $f \in A(\overline{\mathbb{D}})$ and let $E \subset \mathbb{C} \setminus \mathbb{D}$. If $E \cap \mathbb{T} \ne \emptyset$ and if a subsequence of $(s_n f)$ converges pointwise on E to a function h with $h(\zeta) \ne f(\zeta)$, for some $\zeta \in E \cap \mathbb{T}$, then $E \cap \mathbb{D}^*$ has vanishing logarithmic capacity.*

Proof. According to the assumptions, f has radius of convergence 1. Suppose that $E \cap \mathbb{D}^*$ has positive logarithmic capacity. Then the set $\overline{\mathbb{D}} \cup E$ has logarithmic capacity > 1. Let (n_k) be a sequence of integers such that $(s_{n_k} f)_k$ converges to a function h pointwise on E. From Theorem 1 in [4] it follows that f has Hadamard-Ostrowski gaps relative to (n_k). Thus, Lemma 3.1 implies that $h(\zeta) = f(\zeta)$ for all $\zeta \in E \cap \mathbb{T}$. □

In contrast, we have

Theorem 3.3. *Let $E \subset \mathbb{C} \setminus \mathbb{D}$ be a countable set. Then, for each infinite set $\Lambda \subset \mathbb{N}_0$, generically many $f \in A(\overline{\mathbb{D}})$ enjoy the property that for each function $h : E \to \mathbb{C}$ there is a subsequence of $(s_n f)_{n \in \Lambda}$ converging pointwise to h on E.*

This result is stated in [19] for sets $E \subset \mathbb{T}$ (and for $\Lambda = \mathbb{N}_0$). The proof is based on Landau's functions. We give an alternative proof, also for the more general case of $E \subset \mathbb{C} \setminus \mathbb{D}$, based on the Fejér polynomials F_n given by

$$F_n(z) := \left(\frac{1}{n} + \frac{z}{n-1} + \ldots + \frac{z^{n-1}}{1}\right) - \left(\frac{z^{n+1}}{1} + \frac{z^{n+2}}{2} + \ldots + \frac{z^{2n}}{n}\right).$$

The Fejér polynomials have the property that the uniform norms $||F_n||_{\mathbb{T}}$ are bounded, that is,

$$\sup_{n \in \mathbb{N}} \|F_n\|_{\mathbb{T}} < \infty\,,$$

while $s_n F_n(1) \to \infty$ as $n \to \infty$. An elementary proof of the boundedness property of the uniform norms is found in [13].

Of course, different to the situation in the preceding section, we cannot expect any result on simultaneous approximation by polynomials in $A(\overline{\mathbb{D}}) \oplus C(E)$, even for finite sets E touching $\mathbb{T}$. If we allow approximation by polynomials P and partial sums $s_n P$, the situation is more favourable.

Theorem 3.4. *Let $E \subset \mathbb{C} \setminus \mathbb{D}$ be a finite set and $\Lambda \subset \mathbb{N}_0$ an infinite set. Then for each $c \in \mathbb{C}^E$, each polynomial Q and each $\varepsilon > 0$, a polynomial P and a positive integer $n \in \Lambda$ exist with*

$$\|Q - P\|_{\overline{\mathbb{D}}} < \varepsilon \quad \textit{and} \quad s_n P|E = c.$$

Proof. 1. We first prove: If $B \subset \mathbb{C} \setminus \{1\}$ is finite with cardinality m, then a sequence (P_n) of polynomials exists with $\|P_n\|_{\overline{\mathbb{D}}} \to 0$, as $n \to \infty$, and

$$s_n P_n|B = 0, \qquad (s_n P_n)(1) = 1 \qquad (n > m).$$

With F_n the n-th Fejér polynomial we define

$$F_{n,m}(z) := F_n(z) - \sum_{\nu=n-m}^{n-1} \frac{z^\nu}{n-\nu} = \sum_{\nu=0}^{n-m-1} \frac{z^\nu}{n-\nu} - \sum_{\nu=n+1}^{2n} \frac{z^\nu}{\nu-n}$$

for $z \in \mathbb{C}$ and $n > m$. Since $(||F_n||_{\overline{\mathbb{D}}})_n$ is bounded, also $(\|F_{n,m}\|_{\overline{\mathbb{D}}})_{n>m}$ is bounded. Moreover, we have

$$(s_n F_{n,m})(1) = \sum_{k=m+1}^{n} \frac{1}{k} \to \infty \quad (n \to \infty).$$

We define $Q_B(z) := \prod_{w \in B} (z - w)$ for $z \in \mathbb{C}$ and

$$P_n := \frac{1}{Q_B(1)(s_n F_{n,m})(1)} \cdot Q_B \cdot F_{n,m} \quad (n > m).$$

Then $\|P_n\|_{\overline{\mathbb{D}}} \to 0$, as $n \to \infty$. Since $F_{n,m}$ does not contain any of the powers $z^{n-m}, \ldots, z^n$, we further obtain

$$s_n P_n = \frac{Q_B}{Q_B(1)(s_n F_{n,m})(1)} \cdot s_n F_{n,m}$$

and thus $s_nP_n|_B = 0$ as well as $(s_nP_n)(1) = 1$ for all $n > m$.

2. We put $d := c - Q|E$. If

$$0 < \delta < \big(\sum_{w\in E} |d(w)|\big)^{-1}\varepsilon,$$

then part 1 of the proof, applied to $B_w := w^{-1}(E \setminus \{w\})$ for $w \in E$, shows that an integer $n \in \Lambda$ with $n \geq \deg Q$ and polynomials $P_w = P_{n,w}$, for $w \in E$, exist with $\|P_w\|_{\overline{\mathbb{D}}} < \delta$ and

$$s_nP_w|B_w = 0, \qquad (s_nP_w)(1) = 1.$$

Since $|w| \geq 1$, we obtain that $Q_w := P_w(\cdot/w)$ also satisfies $\|Q_w\|_{\overline{\mathbb{D}}} < \delta$ and in addition

$$s_nQ_w|(E \setminus \{w\}) = 0, \qquad (s_nQ_w)(w) = 1.$$

If we define

$$P := Q + \sum_{w\in E} d(w) \cdot Q_w,$$

then

$$\|P - Q\|_{\overline{\mathbb{D}}} \leq \sum_{w\in E} |d(w)|\, \|Q_w\|_{\overline{\mathbb{D}}} < \varepsilon$$

and from $n \geq \deg Q$ we get $s_nQ = Q$. Hence, we have

$$(s_nP)(w) = Q(w) + \sum_{w'\in E} d(w')s_nQ_{w'}(w) = Q(w) + d(w) = c(w)$$

for all $w \in E$. □

Proof of Theorem 3.3. Let first $E \subset \mathbb{C} \setminus \mathbb{D}$ be a finite set. We consider the sequence $(T_n)_{n\in\Lambda}$ of (continuous) linear mappings $T_n : A(\overline{\mathbb{D}}) \to \mathbb{C}^E$, defined by

$$T_nf := s_nf|E \quad (f \in A(\overline{\mathbb{D}})).$$

According to Fejér's theorem, the polynomials are dense in $A(\overline{\mathbb{D}})$. Therefore, the polynomial Q in Theorem 3.4 may be replaced by an arbitrary function $g \in A(\overline{\mathbb{D}})$. According to the Universality Criterion (note that $A(\overline{\mathbb{D}})$ is separable), for generically many functions $f \in A(\overline{\mathbb{D}})$ the set of partial sums $\{s_nf|E : n \in \Lambda\}$ is dense in $\mathbb{C}^E$.

If $E = \{\zeta_j : j \in \mathbb{N}\}$ is countable and if $E_k := \{\zeta_1 \ldots, \zeta_k\}$, then Baire's theorem implies that also generically many functions $f \in A(\overline{\mathbb{D}})$ have the property that for all k the set of partial sums $\{s_nf|E_k : n \in \Lambda\}$ is dense in $\mathbb{C}^{E_k}$. For such f and arbitrary $h : E \to \mathbb{C}$, a strictly increasing sequence (n_j) in Λ exists with

$$|s_{n_j}f(\zeta) - h(\zeta)| < 1/j \qquad (\zeta \in E_j).$$

Then $s_{n_j}f \to h$, as $j \to \infty$, pointwise on E. □

In view of Theorems 3.2 and 3.3, one might ask about compact sets $E \subset \mathbb{C}\backslash\mathbb{D}$ with $E \cap \mathbb{T} \neq \emptyset$ and functions $f \in A(\overline{\mathbb{D}})$ having the property that for each $h \in A(E)$ there exists a subsequence of $(s_n f)$ converging uniformly to h on E.

In [7] (cf. also [25]) it is shown that sets $E \subset \mathbb{T}$ having the property that a function $f \in A(\overline{\mathbb{D}})$ exists with $\{s_n f|E : n \in \mathbb{N}\}$ being dense in $C(E)$ necessarily have to be small in the sense of porosity. In particular, there are countable compact sets $E \subset \mathbb{T}$ with only one accumulation point and so that for no function f in the disc algebra the partial sums $s_n f$ form a dense set in $C(E)$. Actually, simple examples of sets E having this property are given by $E = \{e^{2\pi i/q^k} : k \in \mathbb{N}\}$, for arbitrary $q > 1$. In case of compact sets E lying outside the open unit disc and having an accumulation point on $\mathbb{T}$ the following result follows from [22, Corollary 3.3] and Fejér's theorem:

Proposition 3.5. *Let $E \subset \mathbb{C}\backslash\mathbb{D}$ compact and let $\zeta \in E \cap \mathbb{T}$ such that E contains a sequence (z_n) with $z_n = \zeta(1 + 1/n)e^{i\alpha_n}$ and $\alpha_n = O(1/n)$. Then there does not exist an $f \in A(\overline{\mathbb{D}})$ such that the set $\{s_n f|E : n \in \mathbb{N}_0\}$ is dense in $A(E)$.*

Proposition 3.5 differs from Corollary 3.3 in [22] only in view of the fact that it provides a sufficient condition concerning the speed of convergence of the radii of the sequence (z_n) in order to obtain non-universality of the Taylor series of f on E.

We will show that, on the other hand, uniform universality happens generically. In order to formulate the corresponding results, we introduce the following notation: for a metric space X we denote by $\mathcal{K}(X)$ the metric space of all nonempty compact subsets of X endowed with the Hausdorff metric. It is well-known that $\mathcal{K}(X)$ is complete whenever X is complete (see e.g. [12], Section 2.4).

Theorem 3.6. *For each closed set $B \subset \mathbb{C} \setminus \mathbb{D}$ and each infinite set $\Lambda \subset \mathbb{N}_0$, generically many $f \in A(\overline{\mathbb{D}})$ enjoy the property that $\{s_n f|E : n \in \Lambda\}$ is dense in $C(E)$ for generically many $E \in \mathcal{K}(B)$.*

Proof. The proof will run similarly to the proof of Lemma 2 in [24]. We fix a countable dense subset C of B and we denote $\mathcal{E} := \{E \subset C : E \text{ finite}\}$. Then $\mathcal{E}$ is countable and dense in $\mathcal{K}(B)$. For $f \in A(\overline{\mathbb{D}})$, we define

$$\mathcal{K}_f := \Big\{E \in \mathcal{K}(B) : \{s_n f|E : n \in \Lambda\} \text{ dense in } C(E)\Big\}.$$

Denoting by $\mathcal{P}$ the set of all complex-valued polynomials in two real variables with Gaußian rational coefficients, the complex Stone-Weierstraß theorem implies that the set $\{P|K : P \in \mathcal{P}\}$ is dense in $C(K)$ for each $K \in \mathcal{K}(\mathbb{C})$. Hence, we obtain

$$\mathcal{K}_f = \bigcap_{\substack{j\in\mathbb{N}\\ P\in\mathcal{P}}} \bigcup_{n\in\Lambda} \left\{E \in \mathcal{K}(B) : \|s_n f - P\|_E < \frac{1}{j}\right\}.$$

As each function $s_n f - P$ is uniformly continuous on E, it follows that each set $\{E \in \mathcal{K}(B) : \|s_n f - P\|_E < 1/j\}$ is open in $\mathcal{K}(B)$ so that $\mathcal{K}_f$ is a G_δ-set in $\mathcal{K}(B)$. Due to Theorem 3.3, there exists a residual set $\mathcal{R}$ in $A(\overline{\mathbb{D}})$ such that $\mathcal{E} \subset \mathcal{K}_f$ for all $f \in \mathcal{R}$. Thus, for generically many $f \in A(\overline{\mathbb{D}})$, the set $\mathcal{K}_f$ is a dense G_δ-set and hence in particular residual in $\mathcal{K}(B)$. $\square$

Remark 3.7. For fixed $R > 1$ we consider $B = \mathbb{T} \cup \{z : |z| \geq R\}$. It is known that generically many sets in $\mathcal{K}(B)$ are Cantor sets, i.e. perfect and totally disconnected (cf. [4], Remark 2 on p. 236). Thus, for generically many $f \in A(\overline{\mathbb{D}})$ Theorem 3.6 guarantees the existence of compact sets $E \subset \mathbb{C} \setminus \mathbb{D}$ with both $E \cap \mathbb{T}$ and $E \cap \mathbb{D}^*$ being uncountable and such that $\{s_n f|E : n \in \mathbb{N}_0\}$ is dense in $C(E)$. According to Theorem 3.2, for such sets we necessarily have $E^0 = \emptyset$ (and then $C(E) = A(E)$).

Since [21, Theorem 4.2] shows that generically many functions in the disc algebra also satisfy the property that $\{s_n f|E : n \in \mathbb{N}_0\}$ is dense in $A(E)$ for all compact sets $E \subset \mathbb{D}^*$ with connected complement, it turns out that generically many f enjoy both universality properties simultaneously.

In the statement of Theorem 3.6, the second "generically many"-expression depends on the first one, meaning that the residual subset of $\mathcal{K}(B)$ depends on the choice of a function from the residual subset of $A(\overline{\mathbb{D}})$. This dependence can be interchanged:

Corollary 3.8. *For each closed set $B \subset \mathbb{C} \setminus \mathbb{D}$ and each infinite set $\Lambda \subset \mathbb{N}_0$, generically many $E \in \mathcal{K}(B)$ enjoy the property that $\{s_n f|E : n \in \Lambda\}$ is dense in $C(E)$ for generically many $f \in A(\overline{\mathbb{D}})$.*

Proof. We denote $X := \mathcal{K}(B)$, $Y := A(\overline{\mathbb{D}})$ and

$$S := \big\{(E, f) \in \mathcal{K}(B) \times A(\overline{\mathbb{D}}) : \{s_n f|E : n \in \Lambda\} \text{ dense in } C(E)\big\}.$$

Then X and Y are Baire spaces, Y is second-countable and S is a subset of $X \times Y$. In the following, for $E \in \mathcal{K}(B)$ and $f \in A(\overline{\mathbb{D}})$, we write $S(E, \cdot) := \{f \in A(\overline{\mathbb{D}}) : (E, f) \in S\}$ and $S(\cdot, f) := \{E \in \mathcal{K}(B) : (E, f) \in S\}$. Denoting by $\mathcal{P}$ the set of all complex-valued polynomials in two real variables with Gaußian rational coefficients, the complex Stone-Weierstraß theorem implies, analogously to the proof of Theorem 3.6, that we have

$$\begin{aligned} S(E, \cdot) &= \big\{f \in A(\overline{\mathbb{D}}) : \{s_n f|E : n \in \Lambda\} \text{ dense in } C(E)\big\} \\ &= \bigcap_{\substack{j \in \mathbb{N} \\ P \in \mathcal{P}}} \bigcup_{n \in \Lambda} \left\{ f \in A(\overline{\mathbb{D}}) : \|s_n f - P\|_E < \frac{1}{j} \right\} \end{aligned}$$

for all $E \in \mathcal{K}(B)$. Hence, $S(E, \cdot)$ is a G_δ-set in Y for all $E \in X$. According to Theorem 3.6, we have that

$$S(\cdot, f) = \big\{E \in \mathcal{K}(B) : \{s_n f|E : n \in \Lambda\} \text{ dense in } C(E)\big\}$$

is residual in X for generically many $f \in Y$ (see also the proof of Theorem 3.6, where we have $S(\cdot, f) = \mathcal{K}_f$). Thus, Lemma 3.1 in [7] implies that the set $S(E, \cdot)$ is residual in Y for generically many $E \in X$. $\square$

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