Families of universal Taylor series depending on a parameter

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Abstract

We construct families of universal Taylor series on Ω depending on a parameter $w \in G$, where Ω and G are planar simply connected domains. The functions to be approximated depend on the parameter w, $w \in G$. The partial sums implementing the universal approximation are one variable partial sums with respect to $z \in \Omega$ for each fixed value of the parameter $w \in G$. The universal approximation extends to mixed partial derivatives. This phenomenon is generic in $H(\Omega \times G)$.

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1 Introduction

The first result concerning the existence of universal Taylor series was established before 1914 by Fekete (see [18]). He proved the existence of a real power series $\sum_{n=1}^{\infty} a_n x^n$, whose partial sums approximate uniformly on $[-1, 1]$ every continuous function $h : [-1, 1] \to \mathbb{R}$ with $h(0) = 0$. In the early 1950s Seleznev proved the existence of complex power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence 0, whose partial sums approximate every polynomial uniformly on each compact set $K \subset \mathbb{C} \setminus \{0\}$ with connected complement ([21]). In the early 1970s Luh ([9]) and independently Chui and Parnes ([2]) proved the existence of universal Taylor series with positive radius of convergence defining a function holomorphic in a simply connected domain $\Omega \subset \mathbb{C}$ and whose partial sums approximate every polynomial uniformly on each compact set $K \subset \mathbb{C}$ with connected complement such that $K \cap \overline{\Omega} = \emptyset$.

In the latter result the universal approximation does not necessarily hold on the boundary of the domain of definition Ω. In 1996 a stronger result was obtained, where the universal approximation was valid on the boundary $\partial\Omega$, as well ([15]). The universal approximation was initially implemented by partial sums of the Taylor expansion of the universal function with respect to a fixed center $\zeta \in \Omega$. However, it was soon realized that the result persists when the center in Ω is varied ([16]). After some years ([12], [14]) it was proved that the class of universal functions remains unchanged, whether the center of expansion in a simply connected domain Ω is varied or not. Thus, possible definitions of universal Taylor series are the following ([16], [12]).

Definition 1.1. Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \to \mathbb{C}$ a holomorphic function.

1. For $\zeta_0 \in \Omega$ fixed, the function f belongs to the class $U(\Omega, \zeta_0)$ if the sequence of the partial sums

$$S_N(f, \zeta_0)(z) = \sum_{j=0}^{N} \frac{f^{(j)}(\zeta_0)}{j!}(z - \zeta_0)^j,$$

$N = 0, 1, 2, \ldots$, of the Taylor development of f with center ζ_0 satisfies the following: For every compact set $K \subset \mathbb{C}, K \cap \Omega = \emptyset$ with connected complement K^c and for every function $h : K \to \mathbb{C}$ continuous on K and holomorphic in K°, there exists a sequence (λ_n) of positive integers such that

$$\sup_{z \in K} \left| S_{\lambda_n}(f, \zeta_0)(z) - h(z) \right| \to 0, \quad \text{as } n \to +\infty.$$

2. The function f belongs to the class $U(\Omega)$, if the partial sums

$$S_N(f, \zeta)(z) = \sum_{j=0}^{N} \frac{f^{(j)}(\zeta)}{j!}(z - \zeta)^j,$$

$\zeta \in \Omega, N = 0, 1, 2, \ldots$ satisfy the following condition: For every compact set $K \subset \mathbb{C} \backslash \Omega$ with connected complement and every function $h : K \to \mathbb{C}$ continuous on K and holomorphic in K°, there exists a sequence (λ_n) of positive integers such that for every compact set $L \subset \Omega$ we have

$$\sup_{\zeta \in L} \sup_{z \in K} \left| S_{\lambda_n}(f, \zeta)(z) - h(z) \right| \to 0 \quad \text{as } n \to +\infty.$$

Obviously $U(\Omega, \zeta_0) \supset U(\Omega)$. Further, if Ω is simply connected, both classes $U(\Omega, \zeta_0)$ and $U(\Omega)$ are G_δ and dense in $H(\Omega)$ endowed with the topology of uniform convergence on compact subsets of Ω ([16], [12]). Actually, in this case $U(\Omega, \zeta_0) = U(\Omega)$ ([14], see also [12]).

In this paper, we consider a parameter $w \in G$, where G is some simply connected domain in $\mathbb{C}$, and for every $w \in G$ we find functions $f(\cdot, w)$ in $U(\Omega)$ having the property that a function $h(\cdot, w)$ defined on a compact set $K \subset \mathbb{C}$ and depending on the parameter $w \in G$ can be approximated by the partial sums of $f(\cdot, w)$ with the same sequence (λ_n) for all $w \in G$. Furthermore, the approximation extends to partial derivatives with respect to the parameter w and to mixed partial derivatives with respect to z and w (cf. [17]). It is possible to consider one fixed center of expansion $b(w)$ for every $w \in \Omega$, which is given by a holomorphic function $b : G \to \Omega$, or one may consider all possible centers

$\zeta \in \Omega$. In the latter case, partial derivatives with respect to ζ are also allowed. In this way, functions f holomorphic in $\Omega \times G$ can be constructed in such a way that for, every fixed $w \in G$, the partial sums implementing the universal approximation are those of the functions of one variable $\Omega \ni z \to f(z,w) \in \mathbb{C}$.

We prove that the corresponding universality phenomenon is generic in the space $H(\Omega \times G)$ of holomorphic functions on $\Omega \times G$ endowed with the topology of uniform convergence on compacta.

2 Main results

Let (μ_n) be a strictly increasing sequence of positive integers and (c_j) a sequence of complex numbers. We say that (c_j) has Ostrowski-gaps relative to (μ_n) if a sequence (q_n) exists with $0 < q_n \to 0$ as $n \to \infty$ and so that

$$\sup_{q_n \mu_n \leq j \leq \mu_n} |c_j|^{1/j} \to 0 \qquad (n \to \infty)$$

(see e.g. [13], cf. also [6, p. 311]). Moreover, if (λ_n) is a sequence of positive integers with $\lambda_n = q_n \mu_n$ as above, we say that the sequence (c_j) has Ostrowski-gaps (λ_n, μ_n).

The starting point of our considerations is the following observation:

Proposition 2.1. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain, $f \in U(\Omega) = U(\Omega, \zeta_0)$, K a compact set in $\mathbb{C} \setminus \Omega$ with connected complement, and $h : K \to \mathbb{C}$ a function continuous on K and holomorphic in K°. Let (λ_n) be a sequence as in Definition 1.1. Then for every fixed $z \in K$ we have*

$$\frac{\partial}{\partial \zeta} S_{\lambda_n}(f,\zeta)(z) \to 0 = \frac{\partial}{\partial \zeta} h(z) \quad \textit{as } n \to +\infty$$

uniformly on compact subsets of Ω. Furthermore, the sequence (λ_n) may be chosen so that in addition for every compact set $L \subset \Omega$ we have

$$\sup_{\zeta \in L} \sup_{z \in K} \left| \frac{\partial}{\partial \zeta} S_{\lambda_n}(f,\zeta)(z) \right| \to 0 \quad \textit{as } n \to +\infty.$$

Proof. a) For fixed $z \in K$ the function $\Omega \ni \zeta \to S_{\lambda_n}(f,\zeta)(z) \in \mathbb{C}$ is holomorphic in Ω. According to Definition 1.1 this sequence of elements of $H(\Omega)$ converges uniformly on compacta to the (with respect to $\zeta \in \Omega$) constant function $h(z)$. Weierstrass' theorem implies that $\frac{\partial}{\partial \zeta} S_{\lambda_n}(f,\zeta)(z) \to \frac{\partial h}{\partial \zeta}(z) = 0$ for each $\zeta \in \Omega$ and even uniformly in each compact subset of Ω. Thus we have

$$\sup_{\zeta \in L} \left| \frac{\partial}{\partial \zeta} S_{\lambda_n}(f,\zeta)(z) \right| \to 0, \quad \text{as } n \to +\infty$$

for every fixed $z \in K$.

b) By a straightforward computation we find

$$\begin{aligned}\frac{\partial}{\partial \zeta} S_{\lambda_n}(f,\zeta)(z) &= S_{\lambda_n}(f',\zeta)(z) - S_{\lambda_n - 1}(f',\zeta)(z) \\ &= \frac{((f')^{(\lambda_n)}(\zeta)}{\lambda_n!}(z-\zeta)^{\lambda_n} =: A_{\lambda_n}(\zeta, z).\end{aligned}$$

It is known ([14], [12], [3]) that for any $f \in U(\Omega,\zeta_0) = U(\Omega)$ the sequence of Taylor coefficients of f' with center ζ_0 has Ostrowski gaps relative to some sequence (μ_n) and that the sequence (λ_n) may be chosen so that $q_n\mu_n = \lambda_n - 1$. Then

$$\Big|\frac{(f')^{(\lambda_n)}(\zeta_0)}{\lambda_n!}\Big|^{1/\lambda_n} \to 0 \quad (n \to \infty)$$

and therefore, since $\lambda_n \to +\infty$, we have $\sup_{z\in K}\big|A_{\lambda_n}(\zeta_0, z)\big| \to 0$, as $n \to +\infty$. It follows that

$$\sup_{z\in K}\big|S_{\lambda_n}(f',\zeta_0)(z) - S_{\lambda_n-1}(f',\zeta_0)(z)\big| \to 0, \quad \text{as } n \to +\infty.$$

Since the sequence of Taylor coefficients of f' with center ζ_0 has Ostrowski gaps (λ_n, μ_n) and $(\lambda_n - 1, \mu_n)$, it follows from [12, Lemma 9.2] that for every compact subset L of Ω we have

$$\sup_{\zeta\in L}\sup_{z\in K}\big|S_{\lambda_n}(f',\zeta_0)(z) - S_{\lambda_n}(f',\zeta)(z)\big| \to 0$$

and

$$\sup_{\zeta\in L}\sup_{z\in K}\big|S_{\lambda_{n-1}}(f',\zeta_0)(z) - S_{\lambda_n-1}(f',\zeta)(z)\big| \to 0$$

as $n \to +\infty$. Putting things together it is easily seen that

$$\sup_{\zeta\in L}\sup_{z\in K}\Big|\frac{\partial}{\partial\zeta} S_{\lambda_n}(f,\zeta)(z)\Big| \to 0, \quad \text{as } n \to +\infty.$$

This completes the proof. $\square$

Let Ω and G be two simply connected domains in $\mathbb{C}$. For $f \in H(\Omega \times G)$ and $w \in G, \zeta \in \Omega$ and $z \in \mathbb{C}$ we denote

$$\widetilde{S}_N(f,w,\zeta)(z) = \sum_{j=0}^{N} \frac{\partial^j f}{\partial u^j}(u,w)\Big|_{u=\zeta} \cdot \frac{1}{j!}(z-\zeta)^j$$

and we consider the following classes of functions.

Definition 2.2. Let $b : G \to \Omega$ be a holomorphic function. The class $U(\Omega, G, b)$ contains all functions $f \in H(\Omega \times G)$ such that the sequence $\widetilde{S}_N(f,w,\zeta)$ satisfies the following:

For every compact set $K \subset \mathbb{C}, K \cap \Omega = \emptyset$, K^c connected, and any holomorphic function h in an open neighborhood of $K \times G$ ($h \in H(K \times G)$), there exists a sequence (λ_n) of positive integers such that for every compact set $F \subset G$

$$\sup_{w \in F} \sup_{z \in K} \left| \widetilde{S}_{\lambda_n}\big(f, w, b(w)\big)(z) - h(z, w) \right| \to 0, \quad \text{as } n \to +\infty.$$

Definition 2.3. The class $U(\Omega, G)$ contains all functions $f \in H(\Omega \times G)$ such that the sequence $\widetilde{S}_N(f, w, \zeta)$ satisfies the following:

For every compact set $K \subset \mathbb{C}, K \cap \Omega = \emptyset$, K^c connected and any $h \in H(K \times G)$, there exists a sequence (λ_n) of positive integers such that for all compact sets $F \subset G$, $L \subset \Omega$

$$\sup_{w \in F} \sup_{\zeta \in L} \sup_{z \in K} \left| \widetilde{S}_{\lambda_n}(f, w, \zeta)(z) - h(z, w) \right| \to 0, \quad \text{as } n \to +\infty.$$

Theorem 2.4. *For all holomorphic functions $b : G \to \Omega$ we have*

$$U(\Omega, G) = U(\Omega, G, b).$$

We need parameter modifications of several known results. For potential theoretic notions as for example that of Green's functions and (non-)thinness, we refer to [19]. Moreover, let $||f||_M$ denote the sup-norm of a bounded function f on M.

Lemma 2.5. *Let $F \subset \mathbb{C}$ compact and let $P_n : \mathbb{C} \times F \to \mathbb{C}$ be continuous and such that $P_n(\cdot, w)$ is a polynomial of degree $\leq \mu_n$. If $E \subset \mathbb{C}$ is closed and non-thin at ∞ with*

$$\limsup_{n \to \infty} (||P_n(z, \cdot)||_F)^{1/\mu_n} \leq 1 \quad \text{for all } z \in E,$$

then

$$\limsup_{n \to \infty} (||P_n||_{M \times F})^{1/\mu_n} \leq 1 \quad \text{for all compact } M \subset \mathbb{C}.$$

For sake of completeness, we sketch the proof which is based on Bernstein's lemma (see e.g. [19, Theorem 5.5.7]) and the following characterization of non-thinness at ∞ in terms of Green's functions (see [13, Lemma 1]):

Let $E \subset \mathbb{C}$ be closed and suppose $E_R := \{w \in E : |w| \leq R\}$ to have positive capacity for $R > 0$ sufficiently large. If D_R denotes the component of $\mathbb{C}_\infty \setminus E_R$ containing ∞ then E is non-thin at ∞ if and only if the Green's functions g_{D_R} for D_R satisfy

$$g_{D_R}(z, \infty) \to 0 \quad \text{as } R \to \infty.$$

In a first step, one can reduce the proof to the case of $\mathbb{C} \setminus E$ having no bounded components (cf. [13], proof of Lemma 1). The functions $v_n : \mathbb{C} \to \mathbb{C}$, defined by

$$v_n(z) := \max\left(\frac{1}{\mu_n} \log ||P_n(z, \cdot)||_F, 0\right) \quad \text{for } z \in \mathbb{C},$$

are subharmonic in $\mathbb{C}$ ([19, Theorem 2.4.7]) and from Bernstein's lemma it can be deduced that

$$\limsup_{n\to\infty} v_n(z) \le g_{D_R}(z,\infty) \quad \text{for } z \in D_R$$

(cf. [13], proof of Lemma 1). Then, from the above characterization of non-thinness at ∞, we obtain that $v_n \to 0$ in $\mathbb{C} \setminus E$, as $n \to \infty$. According to the assumption, this implies $v_n \to 0$ in $\mathbb{C}$, where the convergence turns out to be locally uniformly in $\mathbb{C}$. This is equivalent to the statement of Lemma 2.5.

For $F \subset G$ compact and $j = 0, 1, \dots$ we define

$$a_j(F) := \frac{1}{j!} \sup_{w\in F} \Big| \frac{\partial^j f}{\partial u^j}(u,w)\big|_{u=b(w)} \Big|.$$

As an application of Cauchy's estimates we then get (cf. for example the proof of the Lemma in [3])

Lemma 2.6. *Let $F \subset G$ be compact. If (μ_n) is a sequence of integers with*

$$\limsup_{n\to\infty} \sup_{w\in F} (||\tilde{S}_{\mu_n}(f,w,b(w))||_M)^{1/\mu_n} \le 1$$

for all compact $M \subset \mathbb{C}$, then the sequence $(c_j) = (a_j(F))$ has Ostrowski-gaps relative to (μ_n).

A more sophisticated application of Cauchy's estimates in conjunction with the three circles theorem or the two constants theorem yields

Lemma 2.7. *Suppose that $(a_j(F))$ has Ostrowski-gaps (λ_n, μ_n). Then*

$$\sup_{w\in F} \sup_{\zeta\in L} ||\tilde{S}_{\lambda_n}(f,w,\zeta) - \tilde{S}_{\lambda_n}(f,w,b(w))||_M \to 0 \quad \textit{as } n \to \infty,$$

for all compact $L \subset \Omega$ and all compact $M \subset \mathbb{C}$.

The proof is similar to the proof of Theorem 1 of [10]; see also [12, Lemma 9.2].

Proof of Theorem 2.4. Obviously, we have $U(\Omega, G) \subset U(\Omega, G, b)$. Let $f \in U(\Omega, G, b)$. We show that $f \in U(\Omega, G)$. To this aim consider $h \in H(K \times G)$, where K is as in Definition 2.2, and $F \subset G$ compact. Moreover, suppose (K_n) to be an increasing sequence of compact sets in Ω^c with K_n^c connected, $K \cap K_n = \emptyset$ and so that $E := \bigcup_n K_n$ is closed and non-thin at ∞ (such a sequence exists; cf. Lemma 1 in [14]). We define $g_n : K \cup K_n \to \mathbb{C}$ by

$$g_n(z.w) := \begin{cases} h(z,w), & (z,w) \in K \times G \\ 0, & (z,w) \in K_n \times G \end{cases}.$$

The definition of $U(\Omega, G, b)$ implies that a (strictly increasing) sequence (μ_n) exists with

$$\sup_{w\in F}\sup_{z\in K\cup K_n}|\widetilde{S}_{\mu_n}\big(f,w,b(w)\big)(z)-g_n(z,w)|<1/n$$

for all n. Then

$$P_n(z,w):=\widetilde{S}_{\mu_n}\big(f,w,b(w)\big)(z)$$

satisfies the assumptions of Lemma 2.5. Thus, from Lemma 2.5 and Lemma 2.6 we obtain that $(a_j(F))$ has Ostrowski-gaps (λ_n,μ_n). From the definition of Ostrowski-gaps it follows that

$$\sup_{w\in F}\sup_{z\in K}\big|\widetilde{S}_{\mu_n}(f,w,b(w))(z)-\widetilde{S}_{\lambda_n}(f,w,b(w))(z)\big|\to 0,\quad \text{as } n\to+\infty.$$

But then the equiconvergence property of Lemma 2.7 implies that

$$\sup_{w\in F}\sup_{\zeta\in L}\sup_{z\in K}\big|\widetilde{S}_{\lambda_n}(f,w,\zeta)(z)-h(z,w)\big|\to 0,\quad \text{as } n\to+\infty.$$

This shows that $f\in U(\Omega,G)$. □

Remark 2.8. We consider the class $\widetilde{U}(\Omega,G)$. Its definition is the same as the definition of the class $U(\Omega,G)$ but in addition we require the following: For all compact sets $R\subset\Omega$ and $S\subset G$ we have

$$\sup_{z\in R}\sup_{w\in S}\sup_{\zeta\in R}\big|\widetilde{S}_{\lambda_n}(f,w,\zeta)(z)-f(z,w)\big|\to 0\qquad \text{as } n\to+\infty.$$

As in [14, Corollary 1] it is seen that from Ostrowski's classical results on over-convergence and the above proof of Theorem 2.4 it follows that also

$$\widetilde{U}(\Omega,G)=U(\Omega,G).$$

We shall show that the class $\widetilde{U}(\Omega,G)$ is residual in $H(\Omega\times G)$. Actually, we prove this for a subclass of $U(\Omega,G)$.

Definition 2.9. Let $b:G\to\Omega$ be a holomorphic function. The class $U'(\Omega,G,b)$ contains all functions $f\in H(\Omega\times G)$ such that the sequence $\widetilde{S}_N(f,w,\zeta)$ satisfies the following: For every compact set $K\subset\mathbb{C}, K\cap\Omega=\emptyset$, K^c connected, and any holomorphic function h in an open neighborhood of $K\times G$ ($h\in H(K\times G)$), there exists a sequence (λ_n) of positive integers such that the following holds: For every compact set $F\subset G$ and every differential operator $D_{\alpha_1,\alpha_2}=\frac{\partial^{\alpha_1}}{\partial z^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial w^{\alpha_1}}$, $\alpha_1,\alpha_2\in\{0,1,2,\dots\}$ it holds

$$\sup_{w\in F}\sup_{z\in K}\big|D_{\alpha_1,\alpha_2}\widetilde{S}_{\lambda_n}\big(f,w,b(w)\big)(z)-D_{\alpha_1,\alpha_2}h(z,w)\big|\to 0,\quad \text{as } n\to+\infty.$$

Definition 2.10. The class $U'(\Omega, G)$ contains all functions $f \in H(\Omega \times G)$ such that the sequence $\widetilde{S}_N(f, w, \zeta)$ satisfies the following: For every compact set $K \subset \mathbb{C}, K \cap \Omega = \emptyset$, K^c connected and any $h \in H(K \times G)$, there exists a sequence (λ_n) of positive integers such that the following holds: For all compact sets $F \subset G$, $L \subset \Omega$ and for every differential operator $D_{\alpha_1,\alpha_2,\alpha_3} = \frac{\partial^{\alpha_1}}{\partial z^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial w^{\alpha_2}} \frac{\partial^{\alpha_3}}{\partial \zeta^{\alpha_2}}$, $\alpha_1, \alpha_2, \alpha_3 \in \{0, 1, 2, \ldots\}$, it holds

$$\sup_{w \in F} \sup_{\zeta \in L} \sup_{z \in K} \left| D_{\alpha_1,\alpha_2,\alpha_3} \widetilde{S}_{\lambda_n}(f, w, \zeta)(z) - D_{\alpha_1,\alpha_2,\alpha_3} h(z, w) \right| \to 0, \quad \text{as } n \to +\infty.$$

Proposition 2.11. *For all holomorphic functions $b : G \to \Omega$ we have*

$$U'(\Omega, G) \subset U'(\Omega, G, b).$$

Proof. Let $f \in U'(\Omega, G)$. Then, according to the Proposition 2.1 we have

$$\sup_{w \in F} \sup_{\zeta \in L} \sup_{z \in K} \left| D_{\alpha_1,\alpha_2,\alpha_3} \widetilde{S}_{\lambda_n}(f, w, \zeta)(z) \right| \to 0, \quad \text{as } n \to +\infty,$$

provided that $\alpha_3 \neq 0$.

We choose L compact such that $b(F) \subset L \subset \Omega$. Then,

$$\begin{aligned} & D_{\alpha_1,1} \widetilde{S}_{\lambda_n}\big(f, w, b(w)\big)(z) \\ = \quad & D_{\alpha_1,1,0} \widetilde{S}_{\lambda_n}(f, w, b(w))(z) + D_{\alpha_1,0,1} \widetilde{S}_{\lambda_n}(f, w, b(w))(z) \cdot b'(w) \\ \to \quad & D_{\alpha_1,1,0} h(z, w) + 0 \equiv D_{\alpha_1,1} h(z, w), \end{aligned}$$

as $n \to +\infty$, uniformly on $F \times K$, because b' is bounded on the compact set L containing $b(F)$.

For general α_2 the proof follows by induction. □

Theorem 2.12. *The class $U'(\Omega, G)$ is a residual subset of $H(\Omega \times G)$ endowed with the topology of uniform convergence on compacta.*

Proof. It is known (see e.g. [5]) that polynomials in two variables are dense in the space of holomorphic functions defined on the product of two simply connected planar open sets endowed with the topology of uniform convergence on compacta. Thus, the function h can be taken to be a polynomial of two variables.

For compact sets $K \subset \mathbb{C}, F \subset G, L \subset \Omega$, a polynomial h, I a finite subset of $\{0, 1, 2, \ldots\}^3$, $s \in \{1, 2, \ldots\}$ and $n \in \{0, 1, 2, \ldots\}$ we consider the set $E(K, F, L, h, I, s, n)$ of all $g \in H(\Omega \times G)$ such that

$$\sup_{w \in F} \sup_{\zeta \in L} \sup_{z \in K} \left| D_{\alpha_1,\alpha_2,\alpha_3} \widetilde{S}_n(g, w, \zeta)(z) - D_{\alpha_1,\alpha_2,\alpha_3} h(z, w) \right| < \frac{1}{s}$$

for all $(\alpha_1, \alpha_2, \alpha_3) \in I$.

It is known ([12]) that there exists a sequence $K_m, m = 1, 2, \ldots$, of compact subsets of $\mathbb{C} \setminus \Omega$ with K_m^c connected, such that for every compact set $K \subset \mathbb{C} \setminus \Omega$ with K^c connected there exists $m \in \{1, 2, \ldots\}$ so that $K \subset K_m$.

We also consider F_τ, $\tau = 1, 2, \ldots$ and L_ρ, $\rho = 1, 2, \ldots$, two exhausting families of compact sets in G and Ω, respectively. Since G and Ω are simply connected we may assume that F_τ and L_ρ have connected complements ([20]). Finally, let $h_j, j = 1, 2, \ldots$, be an enumeration of the polynomials in two variables with coefficients in $\mathbb{Q} + i\mathbb{Q}$.

One can easily see that

$$U'(\Omega, G) = \bigcap_{I,m,\tau,\rho,j,s} \bigcup_n E(K_m, F_\tau, L_\rho, h_j, I, s, n)$$

where I varies in the set of finite subsets of $\{0, 1, 2, \ldots\}^3$, which is a denumerable set.

If we show that each $E(K, F, L, h, I, s, n)$ is open in $H(\Omega \times G)$, then it will follow that $U(\Omega, G)$ is a G_δ set. Further, if we show in addition that $\bigcup_n E(K_m, F_\tau, L_\rho, h_j, I, s, n)$ is dense in $H(\Omega \times G)$ for every fixed m, τ, ρ, j, I and s then Baire's Category Theorem would imply that $U(\Omega, G)$ is a dense G_δ subset of the Fréchet space $H(\Omega \times G)$.

We consider compact sets M, M_1, T and T_1, such that $L \subset M^\circ \subset M \subset M_1^\circ \subset M_1 \subset \Omega$ and $F \subset T^\circ \subset T \subset T_1^\circ \subset T_1 \subset G$. Let also V be an open set in $\mathbb{C}$ containing K. We consider another two compact sets S and S_1 such that $K \subset S^\circ \subset S \subset S_1^\circ \subset S_1 \subset V$. Then $\mathrm{dist}(M \times T \times S, (M_1^\circ \times T_1^\circ \times S_1^\circ)^c) > r$ for some $r > 0$. Suppose $g \in E(K, F, L, h, I, s, n)$. We show that each $\varphi \in H(\Omega \times G)$ which is sufficiently (uniformly) close to g on the compact set $M_1 \times T_1 \subset \Omega \times G$ belongs to $E(K, F, L, h, I, s, n)$.

By Cauchy estimates on discs with radius r centered on points of $M \times T \supset M^\circ \times T^\circ$ we conclude that $\widetilde{S}_n(\varphi, w, \zeta)(z)$ and $\widetilde{S}_n(g, w, \zeta)(z)$ are close on the open set $M^\circ \times T^\circ \times S^\circ$ if φ is uniformly close to g on the compact set $M_1 \times T_1$. Since $D_{\alpha_1,\alpha_2,\alpha_3}$ is a continuous operator on $H(M^\circ \times T^\circ \times S^\circ)$ it follows that $D_{\alpha_1,\alpha_2,\alpha_3}\widetilde{S}_n(\varphi, w, \zeta)(z)$ and $D_{\alpha_1,\alpha_2,\alpha_3}\widetilde{S}_n(g, w, \zeta)(z)$ are uniformly close on the compact set $L \times F \times K \subset M^\circ \times T^\circ \times S^\circ$. Therefore, $\varphi \in E(K, F, L, h, I, s, n)$ and this set is open.

Next we will show that the sets $\bigcup_n E(K_m, F_\tau, L_\rho, h_j, I, s, n)$ are dense in $H(\Omega \times G)$.

Let $f \in H(\Omega \times G)$, let $\widetilde{L} \subset \Omega$ a compact set, $\widetilde{F} \subset G$ another compact set and $\varepsilon > 0$. Without loss of generality we may assume that $L_\rho \subset \widetilde{L}$ and that $\widetilde{L}^c$ is connected and $F_\tau \subset \widetilde{F}$. We have to find $n \in \{0, 1, 2, \ldots\}$ and $g \in E(K_m, F_\tau, L_\rho, h_j, I, s, n)$ such that

$$\sup_{z \in \widetilde{L}} \sup_{w \in \widetilde{F}} \left| g(z, w) - f(z, w) \right| < \varepsilon.$$

We consider the sets $\widetilde{L} \times \widetilde{F}$ and $K_m \times F_\tau$. Since $\widetilde{L}$ and K_m are disjoint compact sets in $\mathbb{C}$ with connected complements we can find two disjoint simply connected open sets Ω_1 and Ω_2 such that $\widetilde{L} \subset \Omega_1 \subset \Omega$ and $K_m \subset \Omega_2 \subset \mathbb{C}$. We also recall that the open set G contains $\widetilde{F}$ and G is simply connected. The open sets $\Omega_1 \times G$ and $\Omega_2 \times G$ in $\mathbb{C}^2$ are disjoint and $(\Omega_1 \times G) \cup (\Omega_2 \times G) = (\Omega_1 \cup \Omega_2) \times G$ is a product of two simply connected planar open sets. Therefore, Runge's theorem (see e.g. [5]) can be applied to this set.

We consider the holomorphic function $\varphi : (\Omega_1 \cup \Omega_2) \times G \to \mathbb{C}$ defined by $\varphi(z,w) = f(z,w)$ on $\Omega_1 \times G$ and $\varphi(z,w) = h_j(z,w)$ on $\Omega_2 \times G$. Runge's theorem yields a sequence of polynomials $g_\lambda(z,w), \lambda = 1,2,\dots$ converging to $\varphi(z,w)$ uniformly on each compact set of the open set $(\Omega_1 \cup \Omega_2) \times G$. Weierstrass' theorem implies that $D_{\alpha_1,\alpha_2} g_\lambda(z,w) \underset{\lambda\to\infty}{\longrightarrow} D_{\alpha_1,\alpha_2}\varphi(z,w)$ uniformly on compacta of $(\Omega_1 \cup \Omega_2) \times G$. Thus, we can find λ so that, if we set $g = g_\lambda$, we have

$$\sup_{z\in\widetilde{L}} \sup_{w\in\widetilde{F}} \left|g(z,w) - f(z,w)\right| < \varepsilon$$

and

$$\sup_{w\in F_\tau} \sup_{z\in K_m} \left|D_{\alpha_1,\alpha_2} g(z,w) - D_{\alpha_1,\alpha_2} h(z,w)\right| < 1/s$$

for all α_1, α_2 with $(\alpha_1, \alpha_2, 0) \in I$. Now, since g is a polynomial

$$\widetilde{S}_n(g,w,\zeta)(z) = g(w,z) \qquad \text{for all } \zeta,$$

provided n is bigger than the degree of g. Thus

$$D_{\alpha_1,\alpha_2,0}\widetilde{S}_n(g,w,\zeta)(z) = D_{\alpha_1,\alpha_2} g(z,w)$$

and therefore, since $D_{\alpha_1,\alpha_2,0} h(z,w) = D_{a_1,a_2} h(z,w)$,

$$\sup_{w\in F_\tau} \sup_{z\in K_m} \sup_{\zeta\in L_\rho} \left|D_{\alpha_1,\alpha_2,0}\widetilde{S}_n(g,w,\zeta)(z) - D_{\alpha_1,\alpha_2,0} h(z,w)\right| < \frac{1}{s}.$$

If $\alpha_3 \neq 0$ then $D_{\alpha_1,\alpha_2,\alpha_3}\widetilde{S}_n(g,w,\zeta)(z) = D_{\alpha_1,\alpha_2,\alpha_3} g(z,w) = 0$ as well as $D_{\alpha_1,\alpha_2,\alpha_3} h(z,w) = 0$. It follows that

$$\sup_{w\in F_\tau} \sup_{z\in K_m} \sup_{\zeta\in L_\rho} \left|D_{\alpha_1,\alpha_2,\alpha_3}\widetilde{S}_n(g,w,\zeta)(z) - D_{\alpha_1,\alpha_2,\alpha_3} h(z,w)\right| = 0 < \frac{1}{s}.$$

Therefore $\bigcup_n E(K_m, F_\tau, L_\rho, h_j, I, s, n)$ is open and dense in the complete metrizable space $H(\Omega\times G)$. Baire's theorem yields that their denumerable intersection is also G_δ and dense. This proves that $U'(\Omega, G)$ is G_δ and dense. □

Remark 2.13.

- The classes $U'(\Omega, G, b)$ and $U'(\Omega, G)$ are subsets of $H(\Omega \times G)$. We can consider analogous classes in $A^\infty(\Omega \times G)$ (see also [11], [8]). We remind that a holomorphic function $f \in H(\Omega, G)$ belongs to $A^\infty(\Omega \times G)$, iff $D_{\alpha_1,\alpha_2} f$ extends continuously to $\overline{\Omega \times G}$ for all differential operators $D_{\alpha_1,\alpha_2} = \frac{\partial^{\alpha_1}}{\partial z^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial w^{\alpha_2}}$, $\alpha_1, \alpha_2 \in \{0,1,2,\dots,\}$. The topology of $A^\infty(\Omega \times G)$ is defined by the seminorms

$$\sup_{(z,w)\in\overline{\Omega\times G}, \|(z,w)\|\leq n} \left|D_{\alpha_1,\alpha_2} f(z,w)\right|, \; n, \alpha_1, \alpha_2 \in \{0,1,2,\dots\}.$$

In the new definitions the supremuma with respect to z, w, ζ will be calculated on compact subsets of $\overline{\Omega}, \overline{G}$ and $\overline{\Omega}$ respectively, but the universal approximation will be required on compact subsets $K \times G$, $K \cap \overline{\Omega} \neq \emptyset$, K^c connected only. These new classes will be residual in $A^\infty(\Omega \times G)$. The proof is similar to the proof of Theorem 2.12 mainly because the function g, which is a polynomial, obviously belongs to $A^\infty(\Omega \times G)$.

- In all the above results the set $\Omega \times G$ can be replaced by $\Omega \times G_1 \times \cdots \times G_d$, where $\Omega, G_1, \ldots, G_d$ are planar simply connected domains. The proofs are largely the same, because every function $f \in H(\Omega \times G_1, \times \ldots \times G_d)$ can be approximated uniformly on compacta by polynomials ([5]).

- Consider μ any infinite subset of the set of natural numbers. Then in the definition of the class $U(\Omega, G)$ if we require that $\lambda_j \in \mu$ for all $j = 1, 2, \ldots$, then we find another class $U^\mu(\Omega, G)$. This class is also residual. The proof is similar to the proof of the main result of Theorem 2.12. It suffices to mention two points. First, in the description of the class as intersection of a union, the union this time will be taken only for $n \in \mu$. Second, at the density argument we find a polynomial $g(\cdot, \cdot)$ and then we choose a natural number n greater than the degree of g. Certainly we can choose $n \in \mu$, because μ is an infinite subset of the set of natural numbers. Thus $U^\mu(\Omega, G)$ is also residual and hence dense. This implies in a standard way ([1]) algebraic genericity. That is, $U(\Omega, G) \cup \{0\}$ contains a vector subspace dense in $H(\Omega \times G)$.

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