Cumulants are universal homomorphisms into Hausdorff groups

Lutz Mattner
Universität zu Lübeck

14th February 2004

Abstract

This is a contribution to the theory of sums of independent random variables at an algebraico-analytical level: Let $\mathrm{Prob}_\infty(\mathbb{R})$ denote the convolution semigroup of all probability measures on $\mathbb{R}$ with all moments finite, topologized by polynomially weighted total variation. We prove that the cumulant sequence $\kappa = (\kappa_\ell : \ell \in \mathbb{N})$, regarded as a function from $\mathrm{Prob}_\infty(\mathbb{R})$ into the additive topological group $\mathbb{R}^\infty$ of all real sequences, is universal among continuous homomorphisms from $\mathrm{Prob}_\infty(\mathbb{R})$ into Hausdorff topological groups, in the usual sense that every other such homomorphism factorizes uniquely through κ.

An analogous result, referring to just the first $r \in \mathbb{N}_0$ cumulants, holds for the semigroup $\mathrm{Prob}_r(\mathbb{R})$ of all probability measures with existing rth moments.

In particular, there is no nontrivial continuous homomorphism from $\mathrm{Prob}(\mathbb{R})$, the convolution semigroup of all probability measures, topologized by the total variation metric, into any Hausdorff topological group.

1 Introduction and result

Throughout this paper, let $r \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ or $r = \infty$. If r is finite, let $\mathrm{Prob}_r(\mathbb{R})$ denote the set of all probability measures on $\mathbb{R}$ with existing rth moments, topologized by the weighted total variation metric

$$(1) \qquad d_r(P,Q) \;:=\; \int (1+|x|)^r \, d|P-Q|(x) \qquad \big(P, Q \in \mathrm{Prob}_r(\mathbb{R})\big).$$

In particular, $\mathrm{Prob}_0(\mathbb{R})$ is just the set of all probability measures on $\mathbb{R}$, topologized by the ordinary total variation metric. If $r = \infty$, let $\mathrm{Prob}_r(\mathbb{R}) = \mathrm{Prob}_\infty(\mathbb{R}) = \bigcap_{\ell \in \mathbb{N}_0} \mathrm{Prob}_\ell(\mathbb{R})$ denote the set of all probability measures on $\mathbb{R}$ with all moments finite, topologized by the family of metrics $(d_\ell : \ell \in \mathbb{N}_0)$. Then each $\mathrm{Prob}_r(\mathbb{R})$ is a topological semigroup with respect to convolution, compare 2.4 below. What are the continuous homomorphisms from $\mathrm{Prob}_r(\mathbb{R})$ into Hausdorff topological groups? A classical one is the sequence of the

[0] *Mathematics Subject Classification 2000:* 60B15, 60E10, 60G50.

[0] *Keywords and phrases:* algebraic probability theory, characteristic functions, quotients of positive definite functions, sums of independent random variables.

first r cumulants, defined as follows. For $\ell \in \mathbb{N} = \{1, 2, 3, \ldots\}$, let $\kappa_\ell : \mathrm{Prob}_\ell(\mathbb{R}) \to \mathbb{R}$ be the function assigning to $P \in \mathrm{Prob}_\ell(\mathbb{R})$ its ℓth cumulant, that is,

$$\kappa_\ell(P) \;:=\; i^{-\ell}\left(D^\ell \log \circ \widehat{P}\right)(0), \tag{2}$$

where $\widehat{P}(t) := \int e^{itx}\, dP(x)$ for $t \in \mathbb{R}$, $\log \circ \widehat{P}$ is defined in some neighbourhood of zero to be the continuous function g with $g(0) = 0$ and $\exp \circ g = \widehat{P}$, and D denotes differentiation. Regard the set $\mathbb{R}^r$ of all real sequences $(a_\ell : 1 \le \ell < r+1)$ of length r as a topological group, with the usual coordinatewise addition and the product topology. Thus $\mathbb{R}^0 = \{()\}$ is just the trivial topological group while $\mathbb{R}^\infty = \mathbb{R}^\mathbb{N}$ consists of all infinite real sequences, in accordance with the convention $\infty + 1 = \infty$. Then

$$\kappa^{(r)} \;:=\; (\kappa_\ell : 1 \le \ell < r+1), \tag{3}$$

regarded as a function from $\mathrm{Prob}_r(\mathbb{R})$ into $\mathbb{R}^r$, is a continuous homomorphism into a Hausdorff topological group. The purpose of this paper is to show that $\kappa^{(r)}$ is universal in the following sense.

1.1 Theorem. *If G is a Hausdorff topological group and if $\varphi : \mathrm{Prob}_r(\mathbb{R}) \to G$ is a continuous homomorphism, then there is a unique continuous homomorphism $\psi : \mathbb{R}^r \to G$ such that $\varphi = \psi \circ \kappa^{(r)}$.*

This is proved in Section 3 below, after the preparatory Section 2. Turning to corollaries now, let us first emphasize the particular case of Theorem 1.1 where $r = 0$.

1.2 Corollary. *There is no nonconstant continuous homomorphism from* $\mathrm{Prob}(\mathbb{R})$ *into any Hausdorff topological group.*

By specializing the group G in Theorem 1.1 to the multiplicative group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ or to the additive group $\mathbb{R}$, we reobtain the main results of Mattner (1999).

1.3 Corollary (Mattner, 1999). *Let G and φ be as in Theorem 1.1.*

(a) *If $G = \mathbb{T}$, then*

$$\varphi(P) \;=\; \exp\big(i \sum_{\ell=1}^{r} c_\ell \kappa_\ell(P)\big) \qquad \big(P \in \mathrm{Prob}_r(\mathbb{R})\big)$$

for some uniquely determined sequence of real numbers $(c_\ell : 1 \le \ell < r+1)$ with finite support.

(b) *If $G = \mathbb{R}$, then*

$$\varphi(P) \;=\; \sum_{\ell=1}^{r} c_\ell \kappa_\ell(P) \qquad \big(P \in \mathrm{Prob}_r(\mathbb{R})\big)$$

with the c_ℓ as in part (a).

Proof of Corollary 1.3. We recall a few well known facts. Let E be $\mathbb{R}^r$, considered as a topological $\mathbb{R}$-vector space with the usual coordinatewise operations and coordinatewise convergence. Then the dual space of E consists of all functionals ψ of the form

$$\psi(x) \;=\; \sum_{\ell=1}^{r} c_\ell x_\ell \qquad (x \in \mathbb{R}^r) \tag{4}$$

with some sequence of real numbers $(c_\ell : 1 \le \ell < r+1)$, uniquely determined by ψ, with finite support, that is, with $\{\ell : c_\ell \neq 0\}$ finite also in case $r = \infty$. This can be proved by elementary arguments, or by applying the Riesz representation theorem about the dual of $\mathcal{C}(\mathcal{X})$ to the particular locally compact Hausdorff space $\mathcal{X} = \{\ell \in \mathbb{N} : 1 \le \ell < r+1\}$.

Every continuous homomorphism from the additive group of a topological $\mathbb{R}$-vector space into the additive group of $\mathbb{R}$ is automatically a linear functional. Hence every continuous homomorphism ψ from $(\mathbb{R}^r, +)$ to $(\mathbb{R}, +)$ is of the form (4), with the c_ℓ as stated.

Also, every continuous homomorphism from the additive group of a topological $\mathbb{R}$-vector space into the circle group $\mathbb{T}$ is of the form $x \mapsto \exp\big(i\psi(x)\big)$ where ψ is a continuous linear functional, uniquely determined by the homomorphism. See Hewitt & Ross (1979, (23.32.a)) for a proof.

By combining the above with Theorem 1.1, we easily obtain Corollary 1.3. ■

1.4 Remarks.

(a) We refer to Mattner (1999) for corollaries to Corollary 1.3, for counterexamples illuminating the continuity assumptions, and for historical notes and related references. The early history of cumulants is described in more detail by Hald (2000).

(b) The initial inspiration for Mattner (1999) and thereby also for the present work came from Ruzsa & Székely (1988). Their book in particular presents an otherwise unpublished theorem of Halász, which essentially is the case $r = 0$ of Corollary 1.3 (a), or, in an equivalent formulation, Corollary 1.2 restricted to locally compact groups instead of general Hausdorff groups. Here “essentially” refers to a more restrictive continuity assumption imposed by Halász, namely continuity with respect to weak convergence rather than convergence in total variation. The present Theorem 1.1 appears to be a desirable generalization of Halász' theorem, and perhaps final as far as probability measures on $\mathbb{R}$ are concerned.

(c) One may wish to extend Theorem 1.1 to more general groups in place of $\mathbb{R}$. The extension to $\mathbb{R}^d$ with $d \in \mathbb{N}$ is immediate, just leading to more indices everywhere. Compact groups are without interest in the present setting of all probability measures satisfying some growth restriction: If G is a compact group, then $U :=$ normalized Haar measure is an absorbing element of $\mathrm{Prob}(G)$, meaning that $U * P = U$ for every $P \in \mathrm{Prob}(G)$, and it follows that every homomorphism from $\mathrm{Prob}(G)$ into a group maps every P to the neutral element. An extension of Theorem 1.1 to all locally compact abelian groups should pose no serious difficulties. The real challenge is to discuss interesting nonabelian and noncompact cases. For example, is there an analogue of Corollary 1.2 for the general linear groups?

2 Auxiliary analytic facts

This section introduces some notation and collects for subsequent use a few more or less standard facts, without any claim to originality.

2.1 Continuous selection of thresholds. *Let $\mathcal{X}$ be a paracompact topological space and let $(f_t : t \in [1, \infty[)$ be a family of $\{0,1\}$-valued functions on $\mathcal{X}$. Then the following conditions are equivalent:*

(i) *For every $x_0 \in \mathcal{X}$, there exists a neighbourhood U of x_0 and a $t_0 \in [1,\infty[$ with*

$$f_t(x) \;=\; 1 \qquad (x \in U,\, t \geq t_0). \tag{5}$$

(ii) *There exists a continuous function $t_0 : \mathcal{X} \to [1,\infty[$ with*

$$f_t(x) \;=\; 1 \qquad \big(x \in \mathcal{X},\, t \geq t_0(x)\big).$$

Remark. We recall that a Hausdorff topological space is paracompact iff every open cover has a locally finite continuous partition of unity subordinated to it, as used in the proof below. In this paper, namely in the proof of 2.4 (d), we will use the above result with “metric” in place of “paracompact topological”, which is valid since every metric space is paracompact. We refer to Bourbaki (1989, Chapter 9, §4) or Engelking (1989, Section 5.1) for appropriate background information.

Proof. (i) ⇒ (ii): Assuming (i), we choose for every $\xi \in \mathcal{X}$ an open neighbourhood U_ξ of ξ and a $t_\xi \in [1,\infty[$ with $f_t(x) = 1$ for $x \in U_\xi$ and $t \geq t_\xi$. Using the paracompactness of $\mathcal{X}$, we get a locally finite continuous partition of unity $(\varphi_\xi : \xi \in \mathcal{X})$ subordinated to the open cover $(U_\xi : \xi \in \mathcal{X})$ of $\mathcal{X}$. This means, we recall, that for each ξ the function φ_ξ is continuous, nonnegative and with support contained in U_ξ, that every $x \in \mathcal{X}$ has a neighbourhood V such that $\varphi_\xi = 0$ on V for all but finitely many $\xi \in \mathcal{X}$, and that $\sum_{\xi\in\mathcal{X}} \varphi_\xi = 1$. Let us put

$$t_0(x) \;:=\; \sum_{\xi\in\mathcal{X}} t_\xi \varphi_\xi(x) \qquad (x \in \mathcal{X}).$$

Then $t_0 : \mathcal{X} \to [1,\infty[$ is continuous. Let $x \in \mathcal{X}$ and $t \geq t_0(x)$. As $\Xi := \{\xi \in \mathcal{X} : \varphi_\xi(x) > 0\}$ is finite, we may pick a $\xi_0 \in \Xi$ with $t_{\xi_0} = \min\{t_\xi : \xi \in \Xi\}$. Then

$$t_0(x) \;\geq\; \sum_{\xi\in\Xi} t_{\xi_0}\varphi_\xi(x) \;=\; t_{\xi_0}\sum_{\xi\in\Xi}\varphi_\xi(x) \;=\; t_{\xi_0}$$

and hence $t \geq t_{\xi_0}$. As also $x \in U_{\xi_0}$, we get $f_t(x) = 1$.

(ii) ⇒ (i): Assume (ii) and let $x_0 \in \mathcal{X}$. Choose a neighbourhood U of x_0 such that $t_0(x) \leq t_0(x_0)+1$ for $x \in U$. Then (5) is true for the present x_0 and U, with $t_0 := t_0(x_0)+1$. ■

2.2 The r_0 notation. Let us recall that always $r \in \mathbb{N}_0\cup\{\infty\}$ in this paper. In what follows, we will meet finite or infinite sequences of complex numbers indexed by integers starting from 1 or starting from 0. To distinguish sets of such sequences conveniently, we use the standard notation

$$\mathbb{C}^r \;=\; \begin{cases} \mathbb{C}^{\{1,\ldots,r\}} & \text{if } r \in \mathbb{N}_0 \\ \mathbb{C}^{\mathbb{N}} & \text{if } r = \infty \end{cases}$$

along with the nonstandard one

$$\mathbb{C}^{r_0} \;:=\; \begin{cases} \mathbb{C}^{\{0,\ldots,r\}} & \text{if } r \in \mathbb{N}_0 \\ \mathbb{C}^{\mathbb{N}_0} & \text{if } r = \infty. \end{cases}$$

2.3 The topological vector space $\mathcal{C}^r(\mathbb{R})$ and the Taylor map. As usual, let $\mathcal{C}^r(\mathbb{R})$ denote the vector space of all $\mathbb{C}$-valued and r times continuously differentiable functions on $\mathbb{R}$, topologized by convergence of each derivative, uniformly on compact sets. See, for example, Rudin (1991). Again as usual, consider $\mathbb{C}^{r_0}$ as a vector space topologized by coordinatewise convergence. Let $T : \mathcal{C}^r(\mathbb{R}) \to \mathbb{C}^{r_0}$ be the map assigning to each $f \in \mathcal{C}^r(\mathbb{R})$ its sequence of Taylor coefficients, that is,

$$(6) \qquad Tf \;=\; T_r f := (f^{(k)}(0) : k \in \mathbb{N}_0,\, k < r+1) \qquad \big(f \in \mathcal{C}^r(\mathbb{R})\big)$$

where $f^{(k)}$ stands for the kth derivative. Obviously, T is a continuous linear operator. We will need the following three further facts about T.

(a) Borel's theorem. *T is surjective.*

(b) Closure of the null germ. *The closure in $\mathcal{C}^r(\mathbb{R})$ of*

$$N_G \;:=\; \{f \in \mathcal{C}^r(\mathbb{R}) \,:\, f|_U = 0 \text{ for some neighbourhood } U \text{ of } 0\}$$

is

$$(7) \qquad N_T \;:=\; \{f \in \mathcal{C}^r(\mathbb{R}) \,:\, Tf = 0\}.$$

(c) Identification of $\mathcal{C}^r(\mathbb{R})$ modulo the null space of the Taylor map. *Abbreviating N_T from (7) by N, let $\pi : \mathcal{C}^r(\mathbb{R}) \to \mathcal{C}^r(\mathbb{R})/N$ be the quotient map. Then*

$$S\big(\pi(f)\big) \;:=\; T(f) \qquad \big(f \in \mathcal{C}^r(\mathbb{R})\big)$$

defines an isomorphism $S : \mathcal{C}^r(\mathbb{R})/N \to \mathbb{C}^{r_0}$ of topological vector spaces.

Proof. (a) This is nontrivial only in the case $r = \infty$, which is treated, for example, in Donoghue (1969) and Trèves (1967).

(b) Trivially $N_G \subset N_T$. Also N_T, being the null space of the continuous operator T, is closed. Hence the closure $\overline{N_G}$ of N_G is contained in N_T.

To prove that $N_T \subset \overline{N_G}$, let $f \in N_T$. Choose $\omega \in \mathcal{C}^\infty(\mathbb{R})$ with supp $\omega \subset [-1,1]$ and

$$(8) \qquad \omega(x) \;=\; 1 \qquad (|x| \le \frac{1}{2}).$$

Put

$$(9) \qquad g_n(x) \;:=\; f(x)\omega(nx) \qquad (x \in \mathbb{R}).$$

Then $g_n \to 0$ in $\mathcal{C}^r(\mathbb{R})$, since for every fixed $k < r+1$, we have

$$\sup_{x\in\mathbb{R}} |g_n^{(k)}(x)| \;=\; \sup_{|x|\le 1/n} \left| \sum_{j=0}^{k} \binom{k}{j} f^{(j)}(x) n^{k-j} \omega^{(k-j)}(nx) \right| \;=\; o(1) \qquad (n \to \infty)$$

because of $\sup_{|x|\le 1/n} |f^{(j)}(x)| = o((\frac{1}{n})^{k-j})$, the latter following from Taylor's formula with $k-j$ derivatives applied to $f^{(j)}$. Now $f_n := f - g_n \in N_G$ by (8) and (9), and $f_n \to f$ in $\mathcal{C}^r(\mathbb{R})$. Hence $f \in \overline{N_G}$.

(c) S is well defined and injective by the obvious chain of equivalences

$$\pi(f) = \pi(g) \;\Leftrightarrow\; f - g \in N \;\Leftrightarrow\; T(f-g) = 0 \;\Leftrightarrow\; Tf = Tg.$$

It is similarly obvious that S is linear. S is surjective since $T = S \circ \pi$ is surjective by Borel's theorem (a). Continuity of S follows from continuity of T and the definition of the quotient topology. Finally, since $\mathcal{C}^r(\mathbb{R})$ is a Fréchet space, so is its quotient $\mathcal{C}^r(\mathbb{R})/N$. Since $\mathbb{C}^{r_0}$ is Fréchet as well, the open mapping theorem yields openness of S. ∎

2.4 The weighted algebras $M^1_r(\mathbb{R})$. Let us recall the classical Banach algebra

$$M^1(\mathbb{R}) \ := \ \text{all bounded complex measures on } \mathbb{R},$$

with convolution as multiplication and with the norm $\|\cdot\|$ defined by

$$\begin{aligned} \|\mu\| \ &:= \ \int 1\, d|\mu| \qquad (\mu \in M^1(\mathbb{R}), \\ |\mu| \ &:= \ \text{total variation measure of } \mu. \end{aligned} \tag{10}$$

By the standard identification of integrable functions with complex measures, $L^1(\mathbb{R})$ is a subalgebra of $M^1(\mathbb{R})$, and the usual norm on $L^1(\mathbb{R})$ is the restriction of the norm $\|\cdot\|$ from (10). We will need the polynomially weighted subalgebras $M^1_r(\mathbb{R})$ of $M^1(\mathbb{R})$, defined for $r \in \mathbb{N}_0 \cup \{\infty\}$ by

$$M^1_r(\mathbb{R}) \ := \ \Big\{\mu \in M^1(\mathbb{R}) \ : \ \int |x|^l \, d|\mu|(x) < \infty \quad (l \in \mathbb{N}_0,\, l < r+1)\Big\}$$

and topologized by the family of norms $(\|\cdot\|_\ell : \ell \in \mathbb{N}_0,\ \ell < r+1)$ with

$$\|\mu\|_\ell \ := \ \int (1+|x|)^\ell \, d|\mu|(x) \qquad \big(\mu \in M^1_r(\mathbb{R})\big).$$

Obviously, $\mathrm{Prob}_r(\mathbb{R})$ is a topological subspace of $M^1_r(\mathbb{R})$. If r is finite, then it follows from the inequality $(1+|x+y|)^r \le (1+|x|)^r(1+|y|)^r$ that $M^1_r(\mathbb{R})$ is indeed an algebra and, together with the norm $\|\cdot\|_r$, in fact a Banach algebra. This implies that $M^1_\infty(\mathbb{R})$ is a Fréchet algebra. In particular, this shows the continuity of the convolution on $\mathrm{Prob}_r(\mathbb{R})$, taken for granted in the introduction. We further note that the map $\mu \mapsto |\mu|$ is continuous on $M^1_r(\mathbb{R})$. Also, if $r \le s$, then $M^1_s(\mathbb{R})$ is continuously embedded in $M^1_r(\mathbb{R})$.

We now turn to the Fourier transform defined on $M^1(\mathbb{R})$, and hence on each of the subsets $M^1_r(\mathbb{R})$, by

$$\widehat{\mu}(t) \ := \ \int e^{itx}\, d\mu(x) \qquad (t \in \mathbb{R}).$$

For $\mu \in M^1_r(\mathbb{R})$, we have $\widehat{\mu} \in \mathcal{C}^r(\mathbb{R})$, and the map $M^1_r(\mathbb{R}) \ni \mu \mapsto \widehat{\mu} \in \mathcal{C}^r(\mathbb{R})$ is continuous. We will need some kind of continuity of the inverse. The following simple and far from optimal fact is sufficient for our present purposes. We consider

$$\mathcal{G} := \big\{ f \in \mathcal{C}^{r+1}(\mathbb{R}) : \operatorname{supp} f \subset [-1,1] \big\} \tag{11}$$

with the topology inherited from $\mathcal{C}^{r+1}(\mathbb{R})$.

$\mathcal{G}$ is contained in the image of $M^1_r(\mathbb{R})$ under the Fourier transform. The restriction to $\mathcal{G}$ of the inverse Fourier transform is continuous.

Proof. The case $r = \infty$ follows from the case where r is finite. The latter case is obvious from the elementary theory of Sobolev spaces as given in Rudin (1991, Chapter 8) or Taylor (1996, Chapter 4): The inclusion map from $\mathcal{G}$ to the Sobolev space $H^{r+1}(\mathbb{R})$ is continuous. The inverse Fourier transform from $H^{r+1}(\mathbb{R})$ to the weighted L^2 space $L^2_{r+1} := L^2((1+|x|)^{2(r+1)}dx)$ is a homeomorphism. Finally, the inclusion map from L^2_{r+1} to $M^1_r(\mathbb{R})$ is continuous by an application of Cauchy-Schwarz:

$$\begin{aligned} \int |h(x)|(1+|x|)^r\, dx \ &= \ \int |h(x)|(1+|x|)^{r+1} \cdot (1+|x|)^{-1}\, dx \\ &\le \ \|h\|_{L^2_{r+1}} \cdot \Big(\int (1+|x|)^{-2}\, dx \Big)^{1/2} \end{aligned}$$

■

2.5 The multiplication operators M_ϱ on $M^1(\mathbb{R})$. We will need in particular the following refinement (d) of the simple property (13) of the multiplication (or scaling) operators M_ϱ defined for $\varrho \in]0,\infty[$ on $M^1(\mathbb{R})$ by $(M_\varrho\mu)(B) := \mu(\frac{1}{\varrho}B)$, for $B \subset \mathbb{R}$ Borel. In the special case of $f \in L^1(\mathbb{R})$, we have of course $(M_\varrho f)(x) = \frac{1}{\varrho}f(\frac{x}{\varrho})$, for $x \in \mathbb{R}$.

(a) *If $r \in \mathbb{N}_0 \cup \{\infty\}$ and if $\mu \in M^1_r(\mathbb{R})$ is absolutely continuous with respect to Lebesgue measure, then the map $]0,\infty[\ni \varrho \mapsto M_\varrho\mu \in M^1_r(\mathbb{R})$ is continuous.*

(b) *Each M_ϱ is an isometric algebra automorphism of $M^1(\mathbb{R})$, with inverse $M_{1/\varrho}$.*

(c) *Let $\mu \in M^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Then*

$$\lim_{\varrho\to 0} (M_\varrho\mu) * g \;=\; \mu(\mathbb{R}) \cdot g \tag{12}$$

$$\lim_{\varrho\to\infty} \big(\mu * M_\varrho g \;-\; \mu(\mathbb{R}) \cdot M_\varrho g\big) \;=\; 0 \tag{13}$$

with respect to the topology of $M^1(\mathbb{R})$.

(d) *Let $g \in L^1(\mathbb{R})$ and let $\varepsilon > 0$. Then there exists a continuous function $\varrho : M^1(\mathbb{R}) \to [1,\infty[$ with*

$$\|\mu * M_{\varrho(\mu)}g - \mu(\mathbb{R}) \cdot M_{\varrho(\mu)}g\| \;<\; \varepsilon \qquad \big(\mu \in M^1(\mathbb{R})\big).$$

Remark. Relation (13) becomes false if $M^1(\mathbb{R})$ and $L^1(\mathbb{R})$ are replaced by $M^1_r(\mathbb{R})$ and $M^1_r(\mathbb{R}) \cap L^1(\mathbb{R})$ with $r \geq 2$.

Proof. (a), (b) Obvious.

(c) Relation (12) is well known. For example, Reiter (1968, page 6) proves the slightly less general result where $L^1(\mathbb{R})$ replaces $M^1(\mathbb{R})$, in different notation. Sketch of proof: The special case where g is continuous with compact support is easy. Since such functions are dense in $L^1(\mathbb{R})$ and since $\|M_\varrho\mu\| = \|\mu\|$ is uniformly bounded in ϱ, the general case follows by approximation.

Relation (13) follows from (12) and from part (b), starting with

$$\mu * M_\varrho g - \mu(\mathbb{R}) \cdot M_\varrho g \;=\; M_\varrho\big((M_{1/\varrho}\mu) * g - \mu(\mathbb{R}) \cdot g\big).$$

(d) For every $\varrho \in]0,\infty[$, the linear map $M^1(\mathbb{R}) \ni \mu \mapsto \mu * M_\varrho g - \mu(\mathbb{R})M_\varrho g =: \Lambda_\varrho\mu$ has norm $\|\Lambda_\varrho\| \leq 2\|g\|$. Thus the functions $(\Lambda_\varrho : \varrho \in]0,\infty[)$ are equicontinuous and, by part (c), converge to 0 pointwise as $\varrho \to \infty$. Hence, for every $\varepsilon > 0$ and every $\mu_0 \in M^1(\mathbb{R})$, there exists a neighbourhood U of μ_0 and a $\varrho_0 \in [1,\infty[$ with $\|\Lambda_\varrho\mu\| < \varepsilon$ for $\mu \in U$ and $\varrho \geq \varrho_0$. An application of 2.1, to $\mathcal{X} := M^1(\mathbb{R})$ and $f_\varrho(\mu) := 1$ if $\|\Lambda_\varrho\mu\| < \varepsilon$ and $f_\varrho(\mu) := 0$ otherwise, yields the claim.

Proof of the remark. Let us consider $\mu := N_{0,1} - \delta$, $g := N_{0,1}$, with $N_{a,b}$ denoting the normal distribution with mean a and variance b. Then $\mu * M_\varrho g = N_{0,\varrho^2+1} - N_{0,\varrho^2}$ and $\mu(\mathbb{R}) = 0$, and hence $\|\mu * M_\varrho g - \mu(\mathbb{R}) \cdot M_\varrho g\|_2 \geq |\int x^2 d\big(\mu * M_\varrho g\big)(x)| = 1$. As $\mu \in M^1_\infty(\mathbb{R})$, this proves the remark for every $r \geq 2$. ∎

3 Proof of the main theorem

We prepare the proof of Theorem 1.1 with the following three lemmas, the crucial one being 3.1. We put for $k \in \mathbb{N}_0 \cup \{\infty\}$

$$\mathcal{C}^k_{\mathrm{herm}}(\mathbb{R}) \;:=\; \Big\{ f \in \mathcal{C}^k(\mathbb{R}) : f \text{ is hermitean, that is, } f(t) = \overline{f(-t)} \quad (t \in \mathbb{R}) \Big\}.$$

3.1 Quotients of characteristic functions. *We assume that* $r \in 2\mathbb{N}_0 \cup \{\infty\} = \{0, 2, 4, \ldots\} \cup \{\infty\}$ *and consider*

$$\mathcal{F} \;:=\; \big\{ f \in \mathcal{C}^{r+1}_{\mathrm{herm}}(\mathbb{R}) : f(0) = 1 \big\}, \tag{14}$$

with the topology inherited from $\mathcal{C}^{r+1}(\mathbb{R})$.

(a) *If* $f \in \mathcal{F}$, *then there exist* $P, Q \in \mathrm{Prob}_r(\mathbb{R})$ *with* $f\widehat{Q} = \widehat{P}$.

(b) Continuous selection in part (a). *There exists a continuous map* $\mathcal{F} \ni f \mapsto (P_f, Q_f) \in \mathrm{Prob}_r(\mathbb{R}) \times \mathrm{Prob}_r(\mathbb{R})$ *such that*

$$f\widehat{Q}_f \;=\; \widehat{P}_f \qquad (f \in \mathcal{F}). \tag{15}$$

In particular, if $f \in \mathcal{F}$ *and if* (f_n) *is a sequence in* $\mathcal{F}$ *converging to* f *in the* $\mathcal{C}^{r+1}(\mathbb{R})$ *topology, then there exist* $P, Q, P_n, Q_n \in \mathrm{Prob}_r(\mathbb{R})$ *with* $P_n \to P$, $Q_n \to Q$, *with respect to the* $\mathrm{Prob}_r(\mathbb{R})$ *topology, and such that* $f\widehat{Q} = \widehat{P}$ *and* $f_n\widehat{Q}_n = \widehat{P}_n$ *for all* n.

Remark. This kind of lemma goes back to Ruzsa & Székely (1983, 1985, 1988), who prove versions of part (a) in the case $r = 0$. They do not require any differentiability assumption on f, but assume instead that f is the Fourier transform of a bounded signed measure. A version of both parts in the case $r = \infty$ is proved in Mattner (1999, subsection 2.5).

Proof. It is enough to prove the first sentence in part (b). We will use notation and facts from 2.4 and 2.5.

Step 1: Preparation. We fix a $T \in \mathrm{Prob}_\infty(\mathbb{R})$ with $\mathrm{supp}\, \widehat{T} \subset [-1, 1]$. We further fix a $\beta \in]0, 1[$ and a continuous map

$$\mathcal{M} \;:=\; \big\{ \mu \in M^1(\mathbb{R}) : \mu \text{ real}, \mu(\mathbb{R}) = 1 \big\} \ni \mu \;\mapsto\; R_\mu \;\in\; \mathrm{Prob}_\infty(\mathbb{R})$$

satisfying, for every $\mu \in \mathcal{M}$,

$$\|(\mu - \delta) * R_\mu\|_0 \;=:\; \alpha_\mu \;<\; \beta \tag{16}$$

and

$$R_\mu^{*2} \;\geq\; \beta R_\mu. \tag{17}$$

Here continuity of the map $\mu \mapsto R_\mu$ refers to the topology on $\mathcal{M}$ inherited from $M^1(\mathbb{R})$ and to our usual topology on $\mathrm{Prob}_\infty(\mathbb{R})$. In order to see that such a choice of β and $\mu \mapsto R_\mu$ is possible, we may consider for each R_μ a uniform distribution $U_{[-\varrho,\varrho]}$ on the interval $[-\varrho, \varrho]$. Then, regardless of the choice of ϱ, we will have (17) with $\beta := 1/2$. We now apply 2.5 (d) to (the density of) $U_{[-1,1]}$ in place of g, to $1/2$ in place of ε, and to $\mu - \delta$ with $\mu \in \mathcal{M}$ in place of μ. Since $(\mu - \delta)(\mathbb{R}) = 0$, the result is the existence of a continuous function $\varrho : \mathcal{M} \to [1, \infty[$ with

$$\|(\mu - \delta) * M_{\varrho(\mu)} U_{[-1,1]}\|_0 \;<\; \frac{1}{2} \qquad (\mu \in \mathcal{M}).$$

Thus we put $R_\mu := M_{\varrho(\mu)}U_{[-1,1]} = U_{[-\varrho(\mu),\varrho(\mu)]}$, which depends continuously on μ by 2.5 (a) with $r = \infty$.

STEP 2: DEFINITION OF A MAP $f \mapsto (P_f, Q_f)$ SATISFYING (15). Let $f \in \mathcal{F}$. Put $f_0 := f\widehat{T}$. Then $f_0 \in \mathcal{F}$. Let μ be the element of $\mathcal{M} \cap M^1_r(\mathbb{R})$ with

$$f_0 = \widehat{\mu} \tag{18}$$

(compare the statement about $\mathcal{G}$ and the Fourier transform in 2.4). We put

$$R \;:=\; R_\mu,\ \alpha := \alpha_\mu, \tag{19}$$

$$S \;:=\; \beta^{-1}|(\mu-\delta) * R|, \tag{20}$$

$$Q_0 \;:=\; (1-\frac{\alpha}{\beta})R^{*2} * \sum_{k=0}^{\infty} S^{*k} \qquad [\text{convergence in } M^1(\mathbb{R})], \tag{21}$$

$$P \;=\; P_f := \mu * Q_0, \tag{22}$$

$$Q \;=\; Q_f := Q_0 * T. \tag{23}$$

Here the geometric series in (21) converges in $M^1(\mathbb{R})$ since, by (16), S is a sub–probability measure with $\|S\|_0 = S(\mathbb{R}) = \alpha/\beta < 1$. It follows that $Q_0 \in \mathrm{Prob}(\mathbb{R})$ and thus $Q \in \mathrm{Prob}(\mathbb{R})$. Also $P(\mathbb{R}) = 1$ and, easily verified,

$$\begin{aligned}(1-\frac{\alpha}{\beta})^{-1}P &= \mu * R^{*2} * \sum_{k=0}^{\infty} S^{*k} \\ &= R^{*2} + R^{*2} * (\mu - \delta + S) * \sum_{k=0}^{\infty} S^{*k},\end{aligned}$$

where, using (17) and (20),

$$\begin{aligned}R^{*2} * (\mu - \delta + S) &\geq R * (R * (\mu-\delta) + \beta S) \\ &\geq 0.\end{aligned}$$

Hence $P \geq 0$ and thus $P \in \mathrm{Prob}(\mathbb{R})$. Using (22) and (23), we get

$$\widehat{P} \;=\; f_0\widehat{Q}_0 \;=\; f\widehat{T}\widehat{Q}_0 \;=\; f\widehat{Q}$$

and hence (15) holds.

We now check that actually $P, Q \in \mathrm{Prob}_r(\mathbb{R})$. By $0 \leq S \leq \beta^{-1}(|\mu| + \delta) * R$ and by $\mu, R \in M^1_r(\mathbb{R})$, we have $S \in M^1_r(\mathbb{R})$. Hence $\widehat{S} \in \mathcal{C}^r(\mathbb{R})$. Since (23) and (21) show that

$$\widehat{Q}(t) \;=\; (1-\frac{\alpha}{\beta}) \cdot (\widehat{R}(t))^2 \cdot (1-\widehat{S}(t))^{-1}\widehat{T}(t) \qquad (t \in \mathbb{R}), \tag{24}$$

and since $\widehat{R}, \widehat{T} \in \mathcal{C}^\infty(\mathbb{R})$, it follows that $\widehat{Q} \in \mathcal{C}^r(\mathbb{R})$.

From (15) we deduce that $\widehat{P}$ is $\mathcal{C}^r$ as well, at least in some neighbourhood of zero. Since P, Q are probability measures and since $r \in 2\mathbb{N}_0 \cup \{\infty\}$, it follows that $P, Q \in \mathrm{Prob}_r(\mathbb{R})$. For this last step compare, for example, Feller (1971, page 528, problem 15).

To sum up, we have constructed a function $f \mapsto (P_f, Q_f)$ having all properties as claimed, except perhaps continuity.

Step 3: Verification of continuity. Still using the notation introduced in Steps 1 and 2, we first observe that the map

$$\mathcal{F} \ \ni \ f \mapsto (\mu, R, \alpha, S) \ \in \ M_r^1(\mathbb{R}) \times \mathrm{Prob}_\infty(\mathbb{R}) \times [0,\infty[\times M_r^1(\mathbb{R}),$$

defined by (18), (19) and (20), is continuous. To verify this, recall the notation $\mathcal{G}$ from (11), observe that $\mathcal{F} \ni f \mapsto f_0 \in \mathcal{G}$ is continuous, then use the continuity of $\mathcal{G} \ni f_0 \mapsto \mu$ as stated in 2.4, recall that $M_r^1(\mathbb{R})$ is continuously embedded in $M^1(\mathbb{R})$, use continuity of $\mathcal{M} \ni \mu \mapsto R_\mu$ from Step 1, and observe that continuity of $M_r^1(\mathbb{R}) \times \mathrm{Prob}_\infty(\mathbb{R}) \ni (\mu, R) \mapsto (\alpha, S) \in [0,\infty[\times M_r^1(\mathbb{R})$ follows from the properties stated in 2.4.

We now observe that the map

$$\begin{aligned} & \mathrm{Prob}_\infty(\mathbb{R}) \times [0,\infty[\times M_r^1(\mathbb{R}) \ni (R,\alpha,S) \\ \mapsto \ \ & Q_0 \in M_r^1(\mathbb{R}) \text{ with the topology inherited from } M^1(\mathbb{R}) \end{aligned}$$

given by (21) is continuous, and hence so is the map

$$\begin{aligned} & M_r^1(\mathbb{R}) \times \mathrm{Prob}_\infty(\mathbb{R}) \times [0,\infty[\times M_r^1(\mathbb{R}) \ni (\mu, R,\alpha,S) \\ \mapsto \ \ & (P,Q) \in \mathrm{Prob}_r(\mathbb{R}) \times \mathrm{Prob}_r(\mathbb{R}) \text{ with topology from } \mathrm{Prob}(\mathbb{R}) \times \mathrm{Prob}(\mathbb{R}) \end{aligned}$$

given by (21), (22) and (23). (The reader might find it helpful to draw a directed graph with vertices $f, \ldots, Q$ and with arrows indicating the various maps under discussion.)

Translating from the above into the language of convergent sequences, we get in particular: If $f_n \to f$ in $\mathcal{F}$, and if $R_n, \alpha_n, S_n, P_n, Q_n$ correspond to f_n in the same way as R, α, S, P, Q correspond to f, then $S_n \to S$ in $M_r^1(\mathbb{R})$ and

$$P_n \to P, \quad Q_n \to Q \qquad \text{with respect to the Prob}(\mathbb{R}) \text{ topology.} \tag{25}$$

Applying (24) to R_n, α_n, S_n, Q_n and then using (15) with f_n, P_n, Q_n, we get convergence of all moments of P_n and Q_n of order $< r+1$ to the corresponding moments of P and Q. Combined with (25), this yields the desired convergence $P_n \to P$, $Q_n \to Q$ with respect to the $\mathrm{Prob}_r(\mathbb{R})$ topology, as is proved in Mattner (1999, Subsection 2.4), explicitly in the case $r = \infty$ and implicitly in the case $r \in 2\mathbb{N}_0$. ∎

We recall the notation $\kappa^{(r)}$ from (3) and put for abbreviation

$$\kappa \ := \ \kappa^{(\infty)}.$$

3.2 Differences of cumulants. *Every $a \in \mathbb{R}^{\mathbb{N}}$ can be written as $a = \kappa(P) - \kappa(Q)$ for some $P, Q \in \mathrm{Prob}_\infty(\mathbb{R})$.*

Proof. Let $a \in \mathbb{R}^{\mathbb{N}}$. By Borel's theorem 2.3(a), there is a function $g \in \mathcal{C}^\infty(\mathbb{R})$ with Taylor expansion $g(t) \sim \sum_{\ell=1}^\infty a_\ell \frac{(it)^\ell}{\ell!}$ as $t \to 0$. We may assume that g is hermitean, since otherwise we could replace $g(t)$ by $\frac{1}{2}\big(g(t) + \overline{g(-t)}\big)$. Applying 3.1 (a) with $r = \infty$ to $f := e^g$, we get $P, Q \in \mathrm{Prob}_\infty(\mathbb{R})$ with $e^g \widehat{Q} = \widehat{P}$. By taking logarithms near zero and by recalling the definition (2), we get $a = \kappa(P) - \kappa(Q)$. ∎

3.3 Uniqueness of ψ. *If G is a group and if $\varphi : \mathrm{Prob}_r(\mathbb{R}) \to G$ is a homomorphism, then there exists at most one homomorphism $\psi : \mathbb{R}^r \to G$ with $\varphi = \psi \circ \kappa^{(r)}$.*

Remark. With no topology and hence no continuity assumption involved, there need not exist any such ψ.

Proof. It suffices to prove the claim and the remark for $r = \infty$, since the remaining cases follow by considering the restriction $\varphi|_{\mathrm{Prob}_\infty(\mathbb{R})}$. So let $r = \infty$.

Let ψ be a homomorphism as stated. Given $a \in \mathbb{R}^{\mathbb{N}}$, choose P, Q according to 3.2. Then, writing the group operation in G multiplicatively, we have

$$\psi(a) \;=\; \psi\big(\kappa(P)-\kappa(Q)\big) \;=\; \psi\big(\kappa(P)\big)^{-1}\Big(\psi\big(\kappa(Q)\big)\Big)^{-1} \;=\; \varphi(P)\big(\varphi(Q)\big)^{-1}$$

and the right hand side does not depend on ψ.

To prove the remark, define a homomorphism on $\mathrm{Prob}_\infty(\mathbb{R})$ into the multiplicative group of all germs at zero of $\mathcal{C}^\infty$ functions f with $f(0) = 1$ by setting $\varphi(P) :=$ germ at zero of $\widehat{P}$. Let P, Q be any examples proving nonuniqueness in the Stieltjes moment problem, that is, we have $P, Q \in \mathrm{Prob}_\infty(\mathbb{R})$ with supp $P \subset [0,\infty[$ and supp $Q \subset [0,\infty[$, $\kappa(P) = \kappa(Q)$, but $P \neq Q$. (The classical example has for P a log-normal distribution, see Feller (1971, page 227).) Then, for every nonempty open interval I in $\mathbb{R}$, we have $\widehat{P}|_I \neq \widehat{Q}|_I$. (Reason: By the support assumption, $\widehat{P}-\widehat{Q}$ extends to a continuous function on the closed half-plane $\{t \in \mathbb{C} : \mathrm{Im}\ t \geq 0\}$, analytic in the interior. By the reflection principle, such a function can't vanish in I without vanishing identically.) Hence $\varphi(P) \neq \varphi(Q)$. Thus there can't exist any function ψ with $\varphi = \psi \circ \kappa$. ∎

3.4 Proof of Theorem 1.1. Let r, φ and G be as in the theorem. Then, by 3.3, there is at most one ψ as in the theorem. It remains to prove existence. We may assume that G is abelian, for otherwise we could replace G by its subgroup generated by $\varphi\big(\mathrm{Prob}_r(\mathbb{R})\big)$, which obviously is abelian. We will use additive notation for the group operations in G. We also recall the Taylor map notation T_r from (6) and apply it more generally to $\mathcal{C}^r$ functions defined merely in some neighbourhood of zero.

Step 1. *If $P, Q \in \mathrm{Prob}_r(\mathbb{R})$ with $\widehat{P} = \widehat{Q}$ in some neighbourhood U of 0, then $\varphi(P) = \varphi(Q)$.*

Proof. Choose $R \in \mathrm{Prob}_\infty(\mathbb{R})$ such that the support of $\widehat{R}$ is contained in U. Then we have $\widehat{R}\widehat{P} = \widehat{R}\widehat{Q}$ everywhere, and hence $R * P = R * Q$. Applying the homomorphism φ yields $\varphi(R) + \varphi(P) = \varphi(R) + \varphi(Q)$ and thus $\varphi(P) = \varphi(Q)$.

Step 2. *Assume that $r \in 2\mathbb{N}_0 \cup \{\infty\}$. If $P, Q \in \mathrm{Prob}_{r+1}(\mathbb{R})$ with $\kappa^{(r+1)}(P) = \kappa^{(r+1)}(Q)$, then $\varphi(P) = \varphi(Q)$.*

Proof. Let $P, Q \in \mathrm{Prob}_{r+1}(\mathbb{R})$ with $\kappa^{(r+1)}(P) = \kappa^{(r+1)}(Q)$. There exists an $f \in \mathcal{C}^{r+1}_{\mathrm{herm}}(\mathbb{R})$ and a neighbourhood U of zero with

$$f\widehat{Q} = \widehat{P} \quad \text{in } U. \tag{26}$$

To construct f and U, we may put $U_0 := \Big\{\widehat{Q} \neq 0\Big\}$ and $f_0 := \widehat{P}/\widehat{Q}$ in U_0, choose $\omega \in \mathcal{C}^\infty_{\mathrm{herm}}(\mathbb{R})$ with support contained in U_0 and with $\omega = 1$ in some neighbourhood U of zero, and define $f := \omega f_0$ in U_0 and $f := 0$ in $\mathbb{R} \setminus U_0$.

Then $T_{r+1}(\log f) = \Big(0, \big(i^\ell(\kappa_\ell(P) - \kappa_\ell(Q)) : 1 \leq \ell < r+2\big)\Big) = 0$ and hence $T_{r+1}(f - 1) = 0$. Using 2.3(b), we find $f_n \in \mathcal{C}^{r+1}_{\mathrm{herm}}(\mathbb{R})$ with $f_n - 1 = 0$ in some neighbourhood of zero U_n and with $f_n \to f$ in the topology of $\mathcal{C}^{r+1}(\mathbb{R})$. By 3.1 (b), there exist $R, S, R_n, S_n \in \mathrm{Prob}_r(\mathbb{R})$ with $R_n \to R$, $S_n \to S$ and $f\widehat{S} = \widehat{R}$, $f_n\widehat{S}_n = \widehat{R}_n$ for all n. Recalling (26), we see that $\widehat{P}\widehat{S} = \widehat{Q}\widehat{R}$ in the neighbourhood of zero $\{f \neq 0\}$, so that Step 1 applied to $P * S$

and $Q * R$ yields $\varphi(P) + \varphi(S) = \varphi(Q) + \varphi(R)$ and hence

$$\begin{aligned}
\varphi(P) - \varphi(Q) &= \varphi(R) - \varphi(S) \\
&= \lim_{n\to\infty} \big(\varphi(R_n) - \varphi(S_n)\big) \qquad \text{[by continuity of } \varphi\text{]} \\
&= \lim_{n\to\infty} 0 \qquad \text{[Step 1 applied to } R_n \text{ and } S_n\text{]} \\
&= 0.
\end{aligned}$$

Step 3. *If $r = \infty$, then there exists a ψ as stated in the theorem.*

Proof. For every $a \in \mathbb{R}^{\mathbb{N}}$ and any $P, Q \in \mathrm{Prob}_\infty(\mathbb{R})$ with $a = \kappa(P) - \kappa(Q)$, we put

$$\psi(a) \;:=\; \varphi(P) - \varphi(Q). \tag{27}$$

We claim that $\psi : \mathbb{R}^{\mathbb{N}} \to G$ is a well-defined continuous homomorphism with $\varphi = \psi \circ \kappa$.

To prove that ψ is well-defined by (27), let $a \in \mathbb{R}^{\mathbb{N}}$. Then we can choose P, Q as above, by 3.2. Suppose we have two choices, that is, $P_1, Q_1, P_2, Q_2 \in \mathrm{Prob}_\infty(\mathbb{R})$ with $\kappa(P_1) - \kappa(Q_1) = a = \kappa(P_2) - \kappa(Q_2)$. Then $\kappa(P_1 * Q_2) = \kappa(P_2 * Q_1)$, so that Step 2 yields $\varphi(P_1 * Q_2) = \varphi(P_2 * Q_1)$, implying $\varphi(P_1) - \varphi(Q_1) = \varphi(P_2) - \varphi(Q_2)$.

ψ is a homomorphism, since for $a, b \in \mathbb{R}^{\mathbb{N}}$ and $P, Q, R, S \in \mathrm{Prob}_\infty(\mathbb{R})$ with $a = \kappa(P) - \kappa(Q)$, $b = \kappa(R) - \kappa(S)$, we have $a + b = \kappa(P * R) - \kappa(Q * S)$, and hence $\psi(a+b) = \varphi(P * R) - \varphi(Q * S) = \varphi(P) - \varphi(Q) + \varphi(R) - \varphi(S) = \psi(a) + \psi(b)$.

To check that $\varphi = \psi \circ \kappa$, let $P \in \mathrm{Prob}_\infty(\mathbb{R})$. Then, by applying (27) to $a := \kappa(P)$ with the present P and with $Q :=$ the neutral element δ of $\mathrm{Prob}_\infty(\mathbb{R})$, we see that $\psi(\kappa(P)) = \varphi(P) - \varphi(\delta) = \varphi(P)$.

Finally, to prove continuity of ψ, let $a, a_n \in \mathbb{R}^{\mathbb{N}}$ with $a_n \to a$. By 2.3 (c) and by general properties of quotients of topological vector spaces, see Rudin (1991, Sections 1.40, 1.41), there are real-valued $g, g_n \in \mathcal{C}^\infty(\mathbb{R})$ with $Tg = (0, a)$, $Tg_n = (0, a_n)$, and $g_n \to g$. We define $f, f_n \in \mathcal{F}$, compare (14), by setting $f(t) := \exp g(it)$ and $f_n(t) := \exp g_n(it)$ for $t \in \mathbb{R}$. Choosing P, Q, P_n, Q_n according to 3.1 (b), we then have $a = \kappa(P) - \kappa(Q)$ and $a_n = \kappa(P_n) - \kappa(Q_n)$, and hence

$$\begin{aligned}
\psi(a_n) &= \varphi(P_n) - \varphi(Q_n) \\
&\to \varphi(P) - \varphi(Q) \\
&= \psi(a)
\end{aligned}$$

Step 4. *If $r \in \mathbb{N}_0$, then there exists a ψ as stated in the theorem.*

Proof. By applying Step 3 to the restriction of φ to $\mathrm{Prob}_\infty(\mathbb{R})$, we get a continuous homomorphism $\psi_\infty : \mathbb{R}^{\mathbb{N}} \to G$ with

$$\varphi|_{\mathrm{Prob}_\infty(\mathbb{R})} = \psi_\infty \circ \kappa. \tag{28}$$

With a view towards applying Step 2, let us denote by s the smallest even integer greater or equal than r, that is,

$$s \;:=\; \begin{cases} r & (r \in 2\mathbb{N}_0) \\ r+1 & (r \in 2\mathbb{N}_0 + 1). \end{cases}$$

Let $x \in \mathbb{R}^{\mathbb{N}}$. By 3.2, there are $P, Q \in \mathrm{Prob}_\infty(\mathbb{R})$ with

$$\kappa(P) - \kappa(Q) \;=\; (\underbrace{0, \ldots, 0}_{s+1}, x_{s+2}, x_{s+3}, \ldots),$$

so that

$$\begin{aligned}\psi_\infty(0,\ldots,0,x_{s+2},x_{s+3},\ldots) &= \psi_\infty\big(\kappa(P)\big)-\psi_\infty\big(\kappa(Q)\big)\\ &= \varphi(P)-\varphi(Q)\\ &= 0,\end{aligned}$$

where the last equality comes from Step 2, with the present s and $\varphi|_{\mathrm{Prob}_s(\mathbb{R})}$ playing the roles of r and φ. Thus

$$\begin{aligned}\psi_\infty(x) &= \psi_\infty(x_1,\ldots,x_{s+1},0,\ldots)+\psi_\infty(0,\ldots,0,x_{s+2},\ldots)\\ &= \psi_\infty(x_1,\ldots,x_{s+1},0,\ldots).\end{aligned}$$

Hence, with the continuous homomorphism $\psi_{s+1}:\mathbb{R}^{s+1}\to G$ defined by

$$\psi_{s+1}(x) \;:=\; \psi_\infty(x_1,\ldots,x_{s+1},0,\ldots) \qquad (x\in\mathbb{R}^{s+1}),$$

(28) yields

$$\varphi|_{\mathrm{Prob}_\infty(\mathbb{R})} \;=\; \psi_{s+1}\circ\kappa^{(s+1)}|_{\mathrm{Prob}_\infty(\mathbb{R})}. \tag{29}$$

As $\mathrm{Prob}_\infty(\mathbb{R})$ is dense in $\mathrm{Prob}_{s+1}(\mathbb{R})$, we deduce from (29) by continuity that

$$\varphi|_{\mathrm{Prob}_{s+1}(\mathbb{R})} \;=\; \psi_{s+1}\circ\kappa^{(s+1)}. \tag{30}$$

To continue, let $c\in\,]0,\infty[$. By Mattner (1999, Lemma 1.7 b)), there is a sequence (P_n) in $\mathrm{Prob}_\infty(\mathbb{R})$ such that, for $n\to\infty$, we have $P_n\to\delta$ in the topology of $\mathrm{Prob}_s(\mathbb{R})$ and hence $\kappa_\ell(P_n)\to\kappa_l(\delta)=0$ for $\ell=1,\ldots,s$, but $\kappa_{s+1}(P_n)\to c$. Consequently we have, for $n\to\infty$,

$$0=\varphi(\delta)\leftarrow\varphi(P_n)=\psi_{s+1}\big(\kappa_1(P_n),\ldots,\kappa_{s+1}(P_n)\big)\to\psi_{s+1}(\underbrace{0,\ldots,0}_{s},c).$$

As ψ_{s+1} is a homomorphism, we can remove the assumption $c>0$ and get

$$\psi_{s+1}(0,\ldots,0,c) \;=\; 0 \qquad (c\in\mathbb{R}). \tag{31}$$

Hence, with the continuous homomorphism $\psi_s:\mathbb{R}^s\to G$ defined by

$$\psi_s(x_1,\ldots,x_s) \;:=\; \psi_{s+1}(x_1,\ldots,x_s,0) \qquad (x\in\mathbb{R}^s),$$

(30) and (31) yield

$$\varphi(P) \;=\; \psi_s\big(\kappa^{(s)}(P)\big) \qquad (P\in\mathrm{Prob}_{s+1}(\mathbb{R})). \tag{32}$$

As $\mathrm{Prob}_{s+1}(\mathbb{R})$ is dense in $\mathrm{Prob}_s(\mathbb{R})$, identity (32) extends by continuity to $P\in\mathrm{Prob}_s(\mathbb{R})$. In the case of $s=r$, we now put $\psi:=\psi_s$. In the remaining case of $s=r+1$, we repeat the above argument based on Mattner (1999, Lemma 1.7 b)), and end up with $\psi(x_1,\ldots,x_r):=\psi_s(x_1,\ldots,x_r,0)$. ■

Acknowledgement. A major part of this research was done while I was a member of the Department of Statistics, University of Leeds. I thank Bill Crawley-Boevey from Leeds for a helpful conversation. Encouraging comments by several people exposed to preliminary versions of the above are also greatfully acknowledged. This applies in particular to some participants of the March 2003 "International Conference on Probability Theory on Algebraic/Topological Structures" at Bommerholz, organized by Michael Voit.

References

BOURBAKI, N. (1989). *General Topology, Chapters 5-10.* Springer, Berlin.

DONOGHUE, W.F. (1969). *Distributions and Fourier Transforms.* Academic Press, New York.

ENGELKING (1989) *General Topology. Revised and completed edition.* Heldermann, Berlin.

FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications. Vol. II, 2nd. Ed.* Wiley, N.Y.

HALD, A. (2000). The early history of the cumulants and the Gram-Charlier series. *Intern. Statist. Rev.* **68**, 137-153.

HEWITT, E. & ROSS, K.A. (1979). *Abstract Harmonic Analysis I*, 2nd ed. Springer, Berlin.

MATTNER, L. (1999). What are cumulants? *Doc. Math.* **4**, 601-622.
Available at `http://www.mathematik.uni-bielefeld.de/documenta/vol-04/vol-04-eng.html`.

REITER, H. (1968). *Classical Harmonic Analysis and Locally Compact Groups.* Oxford University Press.

RUDIN, W. (1991). *Functional Analysis, 2nd. Ed.* McGraw-Hill, N.Y.

RUZSA, I.Z. & SZÉKELY, G.J. (1983). Convolution quotients of nonnegative functions. *Mh. Mathematik* **95**, 235-239.

RUZSA, I.Z. & SZÉKELY, G.J. (1985). No distribution is prime. *Z. Wahrscheinlichkeitstheorie verw. Geb.* **70**, 263-269.

RUZSA, I.Z. & SZÉKELY, G.J. (1988). *Algebraic Probability Theory.* Wiley, Chichester.

TAYLOR, M.E. (1996). *Partial Differential Equations I. Basic Theory.* Springer, New York.

TRÈVES, F. (1967). *Topological Vector Spaces, Distributions and Kernels.* Academic Press, New York.

UNIVERSITÄT ZU LÜBECK
INSTITUT FÜR MATHEMATIK
WALLSTRASSE 40
23560 LÜBECK
GERMANY
EMAIL: `MattnerL@member.ams.org`
WEB PAGE: www.math.uni-luebeck.de/mattner