On the normal approximation to symmetric binomial distributions

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Abstract

The optimal constant over square root of n error bound in the central limit theorem for distribution functions of sums of independent symmetric Bernoulli random variables is $1/\sqrt{2\pi n}$.

1   Introduction and main result

1.1   Introduction. It has long been known that the error in the central limit theorem for distribution functions of sums of n independent and identically distributed random variables is typically of the order $1/\sqrt{n}$. To be more precise, let P be a probability measure on $\mathbb{R}$ with mean μ, variance $\sigma^2 > 0$, and finite third cumulant $\kappa_3 = \int (x-\mu)^3 \, \mathrm{d}P(x)$. Let h denote the lattice span of P, that is, $h := \sup\{\eta > 0 : \exists\, a \in \mathbb{R} \text{ with } P(a + \eta\mathbb{Z} = 1)\}$ if P is a lattice distribution, and $h := 0$ otherwise. For $n \in \mathbb{N}$, let $F_n = F_{n,P}$ denote the distribution function of the standardized sum of n independent random variables with distribution P, let Φ denote the standard normal distribution function, and let

$$d_n \;=\; d_{n,P} \;=\; \sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)|$$

Then, as proved by Esseen (1956),

$$\lim_{n \to \infty} \sqrt{n}\, d_n \;=\; \frac{1}{6\sqrt{2\pi}} \left(\frac{3h}{\sigma} + \frac{|\kappa_3|}{\sigma^3} \right)$$

Hence, if P is such that $h \neq 0$ or $\kappa_3 \neq 0$, then

$$c_P \;:=\; \sup_{n \in \mathbb{N}} \sqrt{n}\, d_n$$

is a most natural quantity for controlling the approximation errors d_n. We are not aware of a published computation of c_P in any such case. The modest aim of the present paper is to compute c_P in the classical case where $P = \mathrm{B}_{1/2}$ is a symmetric Bernoulli distribution: We show that $c_{\mathrm{B}_{1/2}} = 1/\sqrt{2\pi}$ in Corollary 1.5 below by actually computing d_n for every $n \in \mathbb{N}$ in Corollary 1.4. The heart of the proof is to show in Theorem 1.3 that the supremum in the definition of d_n is attained only in the center of the respective distributions.

1.2   Notation and conventions. Throughout the rest of this paper, let $n \in \mathbb{N}$, let $B_{n,1/2}$ denote the binomial distribution with parameters n and $1/2$, and let for $s \in \mathbb{R}$

$$
\begin{aligned}
F(s) &:= B_{n,1/2}(]-\infty, s]) \\
b_{n,1/2}(s) &:= B_{n,1/2}(\{s\}) = \binom{n}{s} 2^{-s} \\
G(s) &:= \Phi(\frac{s - n/2}{\sqrt{n/4}}) \\
g(s) &:= G(s) - G(s-1)
\end{aligned}
$$

with the understanding that $\binom{n}{s} = 0$ for $s \notin \{0, \dots, n\}$. Φ is the standard normal distribution function with density $\varphi(x) = \Phi'(x) = (1/\sqrt{2\pi})\exp(-x^2/2)$. The left hand limit of a distribution function H at the point s is denoted by $H(s-)$. As usual, $\lfloor x \rfloor := \sup\{s \in \mathbb{Z} : s \le x\}$ and $\lceil x \rceil := \inf\{s \in \mathbb{Z} : s \ge x\}$ for $x \in \mathbb{R}$.

1.3   Theorem. *For every* $n \in \mathbb{N}$

$$
(1) \qquad |F(s) - G(s)| \;<\; F(\lfloor \tfrac{n}{2} \rfloor) - G(\lfloor \tfrac{n}{2} \rfloor) \qquad\qquad (s \in \mathbb{R},\, s \ne \lfloor \tfrac{n}{2} \rfloor)
$$

$$
(2) \qquad |F(s-) - G(s-)| \;<\; G(\lceil \tfrac{n}{2} \rceil -) - F(\lceil \tfrac{n}{2} \rceil -) \qquad (s \in \mathbb{R},\, s \ne \lceil \tfrac{n}{2} \rceil)
$$

1.4   Corollary. *For every* $n \in \mathbb{N}$

$$
(3) \qquad \sup_{s \in \mathbb{R}} \left| B_{n,1/2}(]-\infty, s]) - \Phi(\frac{s - n/2}{\sqrt{n/4}}) \right| = \begin{cases} \Phi(\dfrac{1}{\sqrt{n}}) - \dfrac{1}{2} & (n\ odd) \\[2mm] \dfrac{1}{2} b_{n,1/2}(\dfrac{n}{2}) & (n\ even) \end{cases}
$$

1.5   Corollary. *For every* $n \in \mathbb{N}$

$$
(4) \qquad \frac{\Phi(1) - 1/2}{\sqrt{n}} \;\le\; \sup_{s \in \mathbb{R}} \left| B_{n,1/2}(]-\infty, s]) - \Phi(\frac{s - n/2}{\sqrt{n/4}}) \right| \;<\; \frac{1}{\sqrt{2\pi n}}
$$

The constants $\Phi(1) - 1/2 = 0.3413\dots$ *and* $1/\sqrt{2\pi} = 0.3989\dots$ *are optimal.*

Theorem 1.3 and its corollaries 1.4 and 1.5 are proved in Section 3 using Section 2.

2   Auxiliary analytic inequalities

Given real numbers s, t close to zero, we often need to decide which of the two numbers $(1 + s)/(1 + t)$ and e^{s-t} is larger. If $t > -1$, then the following lemma, applied to either $(x, y) = (s, t)$ or $(x, y) = (t, s)$, yields a decision unless $-t < s < \min(-t + \tfrac{2}{3}t^2, 0)$ or $-s < t < \min(-s + \tfrac{2}{3}s^2, 0)$.

2.1   Lemma. *If* $x, y \in \mathbb{R}$ *satisfy* $0 \le y \le |x|$ *or* $x \le y \le 0$ *or* $x - \tfrac{2}{3}x^2 \ge -y \ge 0$, *then*

$$
(5) \qquad\qquad (1 + x)e^{-x} \;\le\; (1 + y)e^{-y}
$$

holds, and equality occurs iff $x = y$. *The constant* $\tfrac{2}{3}$ *in the assumption can not be replaced by a smaller one.*

Proof. For $t \in \mathbb{R}$,

$$f(t) \quad := \quad (1+t)e^{-t} \quad = \quad 1 - \sum_{n=2}^{\infty} \frac{(-t)^n}{n(n-2)!}$$

satisfies $f'(t) = -te^{-t}$ and $f(t) - f(-t) = 2\sum_{k=1}^{\infty} t^{2k+1}/\big((2k+1)(2k-1)!\big)$. Thus f is strictly increasing on $]-\infty, 0]$, strictly decreasing on $[0, \infty[$, and we have $f(t) - f(-t) > 0$ for $t > 0$. This obviously implies $f(x) \leq f(y)$, which is inequality (5), under each of the first two assumptions, with discussion of equality.

It remains to be proved that the implication

$$(6) \qquad\qquad x - ax^2 \geq -y > 0 \quad \Rightarrow \quad f(x) < f(y)$$

is true if $a = \frac{2}{3}$, and false if $a < \frac{2}{3}$. To this end, we may assume that $\frac{1}{4} < a \leq \frac{2}{3}$. By the isotonicity of f on $]-\infty, 0]$, implication (6) is true iff $x - ax^2 > 0$ implies $f(x) < f(ax^2 - x)$. Let us consider $x \in \mathbb{R}$ with $x - ax^2 \geq 0$, that is, $0 \leq x \leq a^{-1}$. Since $\min_{x \in \mathbb{R}}(1 - x + ax^2) = 1 - \frac{1}{4a} > 0$, we may put

$$h(x) \quad := \quad \log \frac{f(ax^2 - x)}{f(x)} \quad = \quad \log(1 - x + ax^2) - \log(1 + x) + 2x - ax^2$$

and observe that $h(0) = 0$ and

$$(7) \qquad (1 - x + ax^2)(1 + x)h'(x) \quad = \quad (3a - 2)x^2 + (4a - 2a^2)x^3 - 2a^2 x^4$$

If now $a < \frac{2}{3}$, then the right hand side of of (7) is negative for sufficiently small $x > 0$, hence so is $h'(x)$, implying that (6) is false in this case. If, on the other hand, $a = \frac{2}{3}$, then, for all $x > 0$ under consideration, the right hand side of (7) is

$$\frac{8}{9}x^3(2 - x) \quad \geq \quad \frac{8}{9}x^3\Big(2 - \frac{3}{2}\Big) \quad > \quad 0,$$

yielding $h'(x) > 0$ and hence the truth of (6). ∎

2.2   Lemma. *Let $x, y, z \in \mathbb{R}$ with $y \neq z$. Then*

$$(8) \quad \operatorname{sgn}\left(\frac{\Phi(z+x) - \Phi(y+x)}{\Phi(z) - \Phi(y)} - \exp\Big(-\frac{x}{2}(x + y + z)\Big)\right) \quad = \quad \operatorname{sgn}\big(x(x + y + z)\big)$$

and

$$(9) \qquad\qquad \frac{\Phi(z+x) - \Phi(y+x)}{\Phi(z) - \Phi(y)} \quad \leq \quad \exp\Big(-\frac{x^2}{2} - \min(xy, xz)\Big).$$

In particular, if $x \geq 0$, $y < z$ and $x + y + z \geq 0$, then

$$(10) \quad \exp\Big(-\frac{x}{2}(x + y + z)\Big) \quad \leq \quad \frac{\Phi(z+x) - \Phi(y+x)}{\Phi(z) - \Phi(y)} \leq \exp\Big(-\frac{x}{2}(x + 2y)\Big).$$

Proof. For $x, h \in \mathbb{R}$ with $h \neq 0$

$$(11) \qquad\qquad \frac{\Phi(x + \frac{h}{2}) - \Phi(x - \frac{h}{2})}{h\varphi(x)} \quad = \quad \frac{2}{h}\int_0^{h/2} \cosh(xt)e^{-t^2/2}\, dt$$

is obviously a strictly increasing function of $|x|$. Writing $f(x, h)$ for the left hand side of (11), we get, for arbitrary $x, y, z \in \mathbb{R}$ with $y \neq z$,

$$
\begin{aligned}
\mathrm{sgn}\left(\frac{f(\frac{z+y}{2} + x, z - y)}{f(\frac{z+y}{2}, z - y)} - 1\right) &= \mathrm{sgn}\left(|\frac{z+y}{2} + x| - |\frac{z+y}{2}|\right) \\
&= \mathrm{sgn}(x(x + y + z)).
\end{aligned}
$$

Inserting the definition of f yields (8).

To prove (9), we may assume that $y < z$. Then

$$
\begin{aligned}
\Phi(z + x) - \Phi(y + x) &= e^{-x^2/2} \int_y^z e^{-tx} \varphi(t)\, \mathrm{d}t \\
&\leq e^{-x^2/2} e^{-\min(yx, zx)} \int_y^z \varphi(t)\, \mathrm{d}t \\
&= \exp\left(-\frac{x^2}{2} - \min(xy, xz)\right) \cdot \left(\Phi(z) - \Phi(y)\right).
\end{aligned}
$$

∎

2.3   **Lemma.** *For $x > 0$ we have*

$$
(12) \qquad x - \frac{x^3}{6} < xe^{-x^2/6} < \sqrt{2\pi}(\Phi(x) - \frac{1}{2}) < \min\left(x, x - \frac{x^3}{6} + \frac{x^5}{40}\right)
$$

Proof. Put $f(x) := \sqrt{2\pi}(\Phi(x) - \frac{1}{2}) - x\exp(-x^2/6)$. Then $f(0) = 0$ and $f'(x) = e^{-x^2/6}(e^{-x^2/3} - (1 - x^2/3)) > 0$, which implies the central inequality. The others are more obvious. ∎

2.4   **Lemma.** *Let $x, h \in \mathbb{R}$ with $|x| \leq 1$ and $h > 0$. Then*

$$
\Phi(x + \frac{h}{2}) - \Phi(x - \frac{h}{2}) < h\varphi(x).
$$

Proof. We have $\cosh t < e^{t^2/2}$ for $t \neq 0$, by termwise comparison of power series. Hence, using $|x| \leq 1$, the right hand side of (11) is $\leq \frac{2}{h} \int_0^{h/2} \cosh(t) e^{-t^2/2}\, \mathrm{d}t < 1$. ∎

2.5   **Lemma.** *If $k \in \mathbb{N}$, then*

$$
\frac{e^{-\frac{1}{8k}}}{\sqrt{\pi k}} < \mathrm{b}_{2k, \frac{1}{2}}(k) = \mathrm{b}_{2k-1, \frac{1}{2}}(k) = \binom{2k}{k} 2^{-2k} < \frac{1}{\sqrt{\pi k}}.
$$

Proof. The identities are obvious and the right hand inequality is well known. For a proof of the left hand inequality, see Everett (1970, the lower bound in inequality (10)). ∎

3   Proof of the main result

Theorem 1.3 will be proved by combining Propositions 3.2-3.4.

3.1   Lemma. *Let $s \in \mathbb{R}$ with $s \geq n/2$. Then*

$$(13) \qquad \exp\left(-\frac{s - n/2}{n/4}\right) \;\leq\; \frac{g(s+1)}{g(s)} \;\leq\; \exp\left(-\frac{s - n/2 - 1/2}{n/4}\right).$$

Proof. Let $x := 1/\sqrt{n/4}$, $y := (s - 1 - n/2)/\sqrt{n/4}$ and $z := (s - n/2)/\sqrt{n/4}$. Then $x > 0$, $y < z$ and $x + y + z = 2(s - n/2)/\sqrt{n/4} \geq 0$, so that (10) from Lemma 2.2 applies and yields (13). ∎

3.2   Proposition.

$$(14) \qquad \frac{\mathrm{b}_{n,1/2}(s)}{g(s)} \qquad \textit{is strictly decreasing on } \{s \in \mathbb{Z} \,:\, n/2 \leq s \leq n+1\}$$

$$(15) \qquad \mathrm{b}_{n,1/2}(s) \;<\; g(s) \qquad (s \in \mathbb{Z},\, s > \lfloor \tfrac{n}{2} \rfloor)$$

$$(16) \qquad 0 \;<\; F(s) - G(s) \;<\; F(\lfloor \tfrac{n}{2} \rfloor) - G(\lfloor \tfrac{n}{2} \rfloor) \qquad (s \in \mathbb{Z},\, s > \lfloor \tfrac{n}{2} \rfloor)$$

Remark. For $s \in \mathbb{Z}$ with $s \geq n/2 - 1$, the first inequality in (16) is known as part (ii) of Tusnády's Lemma from Bretagnolle & Massart (1989, page 250). Massart (2002, Theorem 1.2) presents the improvement $F(s) \geq \Phi\big((s - \tfrac{n}{2} + \tfrac{1}{4})/\sqrt{n/4}\big)$. Our proof of (16) is independent of these references.

Proof. Claim (14) is equivalent to

$$(17) \qquad \frac{\mathrm{b}_{n,1/2}(s+1)}{\mathrm{b}_{n,1/2}(s)} \;<\; \frac{g(s+1)}{g(s)} \qquad (s \in \mathbb{Z},\, \tfrac{n}{2} \leq s \leq n)$$

So let $s \in \mathbb{Z}$ with $n/2 \leq s \leq n$. Then

$$\begin{aligned}
\frac{\mathrm{b}_{n,1/2}(s+1)}{\mathrm{b}_{n,1/2}(s)} \;&=\; \frac{n-s}{s+1} \;\leq\; \frac{n-s}{s} \;=\; \frac{1 - 2z}{1 + 2z} \qquad \text{with } z := \frac{s - n/2}{n} \\
&\leq\; \exp(-4z) \qquad \text{[by Lemma 2.1 with } x = -2z,\, y = 2z,\, 0 \leq y \leq |x| \,] \\
&=\; \exp\left(-\frac{s - n/2}{n/4}\right)
\end{aligned}$$

and in fact at least one of the above two inequalities is strict. Thus an application of the left hand inequality in (13) from Lemma 3.1 yields (17) and hence (14).

By (14) and by $\mathrm{b}_{n,1/2}(s) = 0$ for $s > n$, it suffices to prove (15) for $s = \lfloor \tfrac{n}{2} \rfloor + 1$.

Let us assume first that $n = 2k$ is even. Then $s = k + 1$. By Lemma 2.3 with $x = 2/\sqrt{n}$, we obtain

$$g(s) \;=\; \Phi(\frac{2}{\sqrt{n}}) - \frac{1}{2} \;>\; \frac{1}{\sqrt{2\pi}}\left(\frac{2}{\sqrt{n}} - \frac{1}{6}\left(\frac{2}{\sqrt{n}}\right)^3\right) \;=\; \frac{1}{\sqrt{\pi k}}\left(1 - \frac{1}{3k}\right)$$

On the other hand, Lemma 2.5 yields

$$\mathrm{b}_{n,1/2}(s) \;=\; \mathrm{b}_{2k,1/2}(k)\frac{k}{k+1} \;<\; \frac{1}{\sqrt{\pi k}}\left(1 - \frac{1}{3k}\right)$$

Thus (15) holds in the present case.

Let us assume now that $n = 2k-1$ is odd. Then $s = k$. By Lemma 2.3 with $x = 1/\sqrt{n}$, we obtain

$$
\begin{aligned}
g(s) \;&=\; \Phi(\frac{1}{\sqrt{n}}) - \Phi(-\frac{1}{\sqrt{n}}) \;=\; 2\left(\Phi(\frac{1}{\sqrt{n}}) - \frac{1}{2}\right) \\
&>\; \frac{2}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{n}} - \frac{1}{6}\left(\frac{1}{\sqrt{n}}\right)^3\right) \\
&=\; \frac{2}{\sqrt{2\pi n}}\left(1 - \frac{1}{6n}\right)
\end{aligned}
$$

On the other hand, Lemma 2.5 yields

$$
\mathrm{b}_{n,1/2}(s) \;=\; \mathrm{b}_{2k-1,1/2}(k) \;<\; \frac{1}{\sqrt{\pi k}} \;=\; \frac{2}{\sqrt{2\pi n}}\sqrt{\frac{2k-1}{2k}}
$$

and we have

$$
\sqrt{\frac{2k-1}{2k}} \;=\; \sqrt{1 - \frac{1}{2k}} \;<\; 1 - \frac{1}{4k} \;<\; 1 - \frac{1}{6n},
$$

proving (15) for the present case as well.

By (15), $F(s) - G(s)$ is strictly decreasing on $\{s \in \mathbb{Z} : s \geq \lfloor n/2 \rfloor\}$. Hence the first inequality in (16) needs only to be proved for $s \geq n$, which cases are trivial since $F(s) = 1$. Also the second inequality is obvious. ∎

3.3   Proposition.

(18) $\dfrac{g(s)}{\mathrm{b}_{n,1/2}(s-1)}$ *is strictly decreasing on* $\left\{s \in \mathbb{Z} : \frac{n+1}{2} \leq s \leq \frac{n}{2} + \sqrt{\frac{3n}{4}} + 1\right\}.$

(19) $g(s) \;<\; \mathrm{b}_{n,1/2}(s-1)$ $\left(s \in \mathbb{Z},\; \lceil \frac{n}{2} \rceil < s \leq \frac{n}{2} + \sqrt{\frac{3n}{4}} + 1\right)$

(20) $G(s) - F(s-1) \;<\; G(\lceil \frac{n}{2} \rceil) - F(\lceil \frac{n}{2} \rceil - 1)$ $\left(s \in \mathbb{Z},\; \lceil \frac{n}{2} \rceil < s \leq \frac{n}{2} + \sqrt{\frac{3n}{4}} + 1\right)$

Proof. For (18), we have to prove

(21) $$\frac{\mathrm{b}_{n,1/2}(s)}{\mathrm{b}_{n,1/2}(s-1)} \;>\; \frac{g(s+1)}{g(s)}$$

under the assumption

(22) $$s \in \mathbb{Z},\qquad \frac{n+1}{2} \leq s \leq \frac{n}{2} + \sqrt{\frac{3n}{4}}$$

So let us assume (22). Then $s \leq n$ (since otherwise we would have $n + 1 \leq \frac{n}{2} + \sqrt{3n/4}$, which is false) and

$$
\frac{\mathrm{b}_{n,1/2}(s)}{\mathrm{b}_{n,1/2}(s-1)} \;=\; \frac{n-s+1}{s} \;=\; \frac{1+y}{1+x}
$$

with

$$x := \frac{s - \frac{n}{2}}{\frac{n}{2}}, \qquad y := -\frac{s - \frac{n}{2} - 1}{\frac{n}{2}}$$

satisfying the assumptions of Lemma 2.1: If $-y \geq 0$, then we observe that

$$x + y - \frac{2}{3}x^2 \;=\; \frac{2}{n} - \frac{2}{3}\left(\frac{s - \frac{n}{2}}{\frac{n}{2}}\right)^2 \;\geq\; 0$$

so that $x - \frac{2}{3}x^2 \geq -y \geq 0$. If, on the other hand, $-y < 0$, then $s = (n+1)/2$ and hence $y = 1/n = x$, so that we have a trivial case where $0 \leq y \leq |x|$. Hence Lemma 2.1 yields

$$\frac{\mathrm{b}_{n,1/2}(s)}{\mathrm{b}_{n,1/2}(s-1)} \;=\; \frac{1+y}{1+x} \;\geq\; e^{y-x} \;=\; \exp\left(-\frac{s - \frac{n}{2} - \frac{1}{2}}{\frac{n}{4}}\right)$$

The right hand inequality in (13) from Lemma 3.1 now yields (21) and hence (18).

By (18), it suffices to prove (19) for $s = \lceil \frac{n}{2} \rceil + 1$.

If $n = 2k$ is even, then $s = k + 1$ and we get

$$
\begin{aligned}
g(s) \;&=\; \Phi(\frac{2}{\sqrt{n}}) - \frac{1}{2} \\
&<\; \frac{1}{\sqrt{2\pi}}\left(\frac{2}{\sqrt{n}} - \frac{1}{6}\left(\frac{2}{\sqrt{n}}\right)^3 + \frac{1}{40}\left(\frac{2}{\sqrt{n}}\right)^5\right) \qquad \text{[by Lemma 2.3]} \\
&=\; \frac{1}{\sqrt{\pi k}}\left(1 - \frac{1}{3k} + \frac{1}{10k^2}\right) \\
&<\; \frac{1}{\sqrt{\pi k}}\left(1 - \frac{1}{8k}\right) \qquad \text{[using } k \geq 1] \\
&<\; \frac{1}{\sqrt{\pi k}}e^{-\frac{1}{8k}} \\
&<\; \mathrm{b}_{2k,1/2}(k) \qquad \text{[by Lemma 2.5]} \\
&=\; \mathrm{b}_{n,1/2}(s-1)
\end{aligned}
$$

If $n = 2k - 1$ is odd, then $s = k + 1$ and we have for $n \notin \{1, 3\}$

$$
\begin{aligned}
g(s) \;&=\; \Phi(\frac{3}{\sqrt{n}}) - \Phi(\frac{1}{\sqrt{n}}) \\
&<\; \frac{2}{\sqrt{n}}\varphi(\frac{2}{\sqrt{n}}) \qquad \text{[by Lemma 2.4 with } x = \frac{2}{\sqrt{n}} \leq 1] \\
&=\; \frac{1}{\sqrt{\pi k}}\sqrt{\frac{k}{k - \frac{1}{2}}}\exp(-\frac{1}{k - \frac{1}{2}}) \\
&<\; \frac{1}{\sqrt{\pi k}}\sqrt{1 + \frac{1}{k}}\exp(-\frac{1}{k}) \\
&<\; \frac{1}{\sqrt{\pi k}}e^{-\frac{1}{2k}} \qquad \text{[by } \sqrt{1+x} < e^{x/2}] \\
&<\; \mathrm{b}_{2k-1,1/2}(k) \qquad \text{[by Lemma 2.5]} \\
&=\; \mathrm{b}_{n,1/2}(s-1)
\end{aligned}
$$

Finally, for $n = 1$ we have

$$g(s) = \Phi(3) - \Phi(1) < \frac{1}{2} = b_{1,1/2}(1) = b_{n,1/2}(s-1)$$

and for $n = 3$ we have

$$g(s) = \Phi(\sqrt{3}) - \Phi(\frac{1}{\sqrt{3}}) = 0.2402\ldots < \frac{3}{8} = b_{3,1/2}(2) = b_{n,1/2}(s-1)$$

which completes the proof of (19).

By (19), $G(s) - F(s-1)$ is strictly decreasing on $\left\{s \in \mathbb{Z} : \frac{n}{2} \le s \le \frac{n}{2} + \sqrt{3n/4} + 1\right\}$. Thus (20) holds. ∎

3.4   Proposition. *For $s \in \mathbb{Z}$ with $(s - 1 - \frac{n}{2})/\sqrt{n/4} \ge \sqrt{3}$, we have*

$$G(s) - F(s-1) \;\; < \;\; \frac{2\exp(-3/2)}{\sqrt{2\pi n}} = \frac{0.1780\ldots}{\sqrt{n}}$$

Proof.

$$
\begin{aligned}
G(s) - F(s-1) \;\;&=\;\; G(s-1) - F(s-1) + G(s) - G(s-1) \\
&<\;\; G(s) - G(s-1) && \text{[by (16)]} \\
&<\;\; \frac{1}{\sqrt{n/4}}\varphi\left(\frac{s-1-\frac{n}{2}}{\sqrt{n/4}}\right) && \text{[as φ decreases strictly on $[0,\infty[$]} \\
&\le\;\; \frac{1}{\sqrt{n/4}}\varphi(\sqrt{3})
\end{aligned}
$$

∎

3.5   Some calculations. We will use the symmetries

(23) $$F(s) \;\;=\;\; 1 - F(n - s - 1) \qquad (s \in \mathbb{Z})$$

(24) $$G(s) \;\;=\;\; 1 - G(n - s) \qquad (s \in \mathbb{R})$$

By (23), we have in particular $F(\lfloor n/2 \rfloor) = F(\lceil n/2 \rceil -) = 1/2$ if n is odd, and $F(\lfloor n/2 \rfloor) = 1/2 + (1/2)b_{n,1/2}(n/2)$ and $F(\lceil n/2 \rceil -) = 1/2 - (1/2)b_{n,1/2}(n/2)$ if n is even. Hence, using (24), we get

(25) $$d_n \;\; := \;\; F(\lfloor \tfrac{n}{2} \rfloor) - G(\lfloor \tfrac{n}{2} \rfloor) = G(\lceil \tfrac{n}{2} \rceil -) - F(\lceil \tfrac{n}{2} \rceil -) = \begin{cases} \Phi(\frac{1}{\sqrt{n}}) - \frac{1}{2} & (n \text{ odd}) \\ \frac{1}{2}b_{n,1/2}(\frac{n}{2}) & (n \text{ even}) \end{cases}$$

Lemmas 2.3 and 2.5 yield

(26) $$d_n \;\; < \;\; \frac{1}{\sqrt{2\pi n}} \qquad (n \in \mathbb{N})$$

(27) $$\lim_{n\to\infty} \sqrt{n}\, d_n \;\; = \;\; \frac{1}{\sqrt{2\pi}}$$

Let us put $a_k := \sqrt{2k-1}\, d_{2k-1}$ and $b_k := \sqrt{2k}\, d_{2k}$ for $k \in \mathbb{N}$. The sequence $(a_k)_{k\in\mathbb{N}}$ is increasing due to the concavity of Φ on $[0,\infty[$. The sequence $(b_k)_{k\in\mathbb{N}}$ is increasing since $b_{k+1}/b_k = (2k+1)/\sqrt{2k(2k+2)} > 1$. As $a_1 = \Phi(1) - 1/2 = 0.3413\ldots < 0.3535\ldots = \sqrt{2}/4 = b_1$, it follows that

(28) $$\sqrt{n}\, d_n \;\; \ge \;\; \sqrt{1}\, d_1 = \Phi(1) - 1/2 = 0.3413\ldots \qquad (n \in \mathbb{N})$$

3.6 Proof of Theorem 1.3. As F is the distribution function of a probability measure concentrated in $\mathbb{Z}$ and as G is continuous and strictly increasing, we have for $s \in \mathbb{R} \setminus \mathbb{Z}$

$$
\begin{aligned}
F(s) - G(s) &= F(s-) - G(s-) < F(\lfloor s \rfloor) - G(\lfloor s \rfloor) \\
G(s) - F(s) &= G(s-) - F(s-) < G(\lceil s \rceil -) - F(\lceil s \rceil -)
\end{aligned}
$$

Hence, taking into account the central equality in (25), we may assume that $s \in \mathbb{Z}$ when proving (1) and (2).

If $s \in \mathbb{Z}$ with $s < \lfloor n/2 \rfloor$, then $t := n - s > \lceil n/2 \rceil$ and, using (23) and (24),

$$
F(s) - G(s) = G(n-s) - F(n-s-1) = G(t-) - F(t-)
$$

Similarly, if $s \in \mathbb{Z}$ with $s < \lceil n/2 \rceil$, then $t := n - s > \lfloor n/2 \rfloor$ and

$$
F(s-) - G(s-) = G(n-s) - F(n-s) = G(t) - F(t)
$$

Hence it is enough to prove (1) and (2) for $s \in \mathbb{Z}$ and with "$\neq$" replaced by "$>$".

If $s \in \mathbb{Z}$ with $s > \lfloor n/2 \rfloor$, then the inequality in (1) is contained in (16).

So it remains to prove the inequality in (2) assuming $s \in \mathbb{Z}$ and $s > \lceil n/2 \rceil$. If $F(s-1) - G(s) > 0$, then

$$
|F(s-) - G(s-)| = F(s-1) - G(s) \le F(s) - G(s) < d_n
$$

by (1) with $s > \lfloor n/2 \rfloor$. So let $|F(s-) - G(s-)| = G(s) - F(s-1)$. If $s \le n/2 + \sqrt{3n/4} + 1$, then $G(s) - F(s-1) < d_n$ by (20). Finally, if $s > n/2 + \sqrt{3n/4} + 1$, then Proposition 3.4 and inequality (28) yield

$$
G(s) - F(s-1) < \frac{2\exp(-3/2)}{\sqrt{2\pi n}} < \frac{\Phi(1) - 1/2}{\sqrt{n}} \le d_n
$$

■

3.7 Proof of Corollary 1.4. Obvious by (1) and (25). ■

3.8 Proof of Corollary 1.5. Obvious by (3), (26), (27) and (28). ■

References

BRETAGNOLLE, J. & MASSART, P. (1989). Hungarian constructions from the nonasymptotic viewpoint. *Ann. Prob.* **17**, 239-256.

ESSEEN, C.G. (1956). A moment inequality with an application to the central limit theorem. *Skandinavisk Aktuarietidskrift* **39**, 160-170.

EVERETT, C.J. (1970). Inequalities for the Wallis product. *Math. Mag.* **43**, 30-33.

MASSART, P. (2002). Tusnady's Lemma, 24 years later. *Ann. Inst. Henri Poincaré, Probab. Stat.* **38**, 991-1007.

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