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Mathematical Notes

Bernstein's theorem, inversion formula of Post and Widder, and the uniqueness theorem for Laplace transforms

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Abstract: A short and natural development of the theorems mentioned in the title is given.

Introduction

Let μ denote a probability measure on the halfline $[0,\infty)$. Then the function $\varphi : [0,\infty) \mapsto \mathbb{R}$ defined by

$$\varphi(t) = \int_{[0,\infty)} e^{-tx} d\mu(x) \tag{1}$$

is called the Laplace transform of μ. What functions φ arise as Laplace transforms of probability measures? If φ is given by (1), then we necessarily have

(i) φ is continuous in $[0,\infty)$ with $\varphi(0) = 1$,

(ii) φ possesses in $(0,\infty)$ derivatives of all orders with $(-1)^n \varphi^{(n)}(t) \geq 0$ $(t > 0,$ $n = 0, 1, \ldots)$.

A celebrated theorem of S. Bernstein states that the above conditions are in turn sufficient for the existence of a probability measure μ such that (1) holds.

Theorem (Bernstein). *Every function $\varphi : [0,\infty) \mapsto \mathbb{R}$ which satisfies (i) and (ii) is the Laplace transform of a probability measure on $[0,\infty)$.*

The principle aim of the paper is to give a short and natural proof of this theorem. It is not significantly longer than Feller's proof (see [5, Chapter XIII.4]) but contains no ad hoc elements and does not presuppose any knowledge on Laplace transforms.

Proof of Bernsteins Theorem

Proof. In addition to (i) and (ii) we may assume

(iii) $\varphi(\infty) := \lim_{t\to\infty} \varphi(t) = 0.$

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In fact, because φ is decreasing and nonnegativ, this limit exists. In case $\varphi(\infty) = 1$, (1) holds with μ being the Dirac measure at zero, whereas if $\varphi(\infty) \in (0,1)$, we can consider $\frac{\varphi-\varphi(\infty)}{1-\varphi(\infty)}$ in place of φ.

Fix a $t > 0$. Since we want a representation of $\varphi(t)$ as an integral, it is natural to look at the Taylor series of φ about a point $A > 0$ with integral remainder for every positive integer n:

$$\varphi(t) = \sum_{k=0}^{n-1} \frac{(-1)^k \varphi^{(k)}(A)}{k!}(A-t)^k + \int_t^A \frac{(s-t)^{n-1}}{(n-1)!}(-1)^{(n)}\varphi^{(n)}(s)\,ds.$$

Let A tend to infinity in the above formula, keeping n fixed. Since the integral, being bounded by $\varphi(t)$ and having a nonnegative integrand by assumption, converges to some finite limit, so does the sum for every n, which implies the convergence of every term. But the limits of these terms are independent of t, because of $\lim_{A\to\infty} \frac{A-t}{A} = 1$. This shows the existence of constants c_n, such that

$$\begin{aligned}\varphi(t) &= c_n + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}(-1)^n \varphi^{(n)}(s)\,ds \\ &= c_n + \int_0^\infty \left(1-\frac{t}{s}\right)_+^{n-1} \frac{(-1)^n}{(n-1)!} s^{n-1}\varphi^{(n)}(s)\,ds \end{aligned} \tag{2}$$

for every $t > 0$ and positive integer n, where in (2) and in what follows, x_+ denotes the greater of x and 0.

Now let t tend to infinity, for fixed $n \geq 2$. By dominated convergence the integral in (2) tends to zero which, together with (iii), implies that the c_n actually vanish.

By the substitution $s = n/x$ we get

$$\varphi(t) = \int_0^\infty \left(1-\frac{tx}{n}\right)_+^{n-1} f_n(x)\,dx \tag{3}$$

for $t > 0$, where

$$f_n(x) := \frac{(-1)^n}{n!}\left(\frac{n}{x}\right)^{n+1} \varphi^{(n)}\left(\frac{n}{x}\right) I(x>0) \tag{4}$$

and $I(x > 0)$ denotes the indicator (or characteristic function) of the set $\{x > 0\}$. By dominated convergence, we see that (3) is valid even for $t = 0$. Thus the f_n are probability densities. According to Helly's selection principle (see [**5**, p. 267]), there is a subsequence of (f_n) such that the sequence of the corresponding measures tends vaguely to a nonnegative measure μ on $[0,\infty)$ with $\mu([0,\infty)) \leq 1$. Together with the elementary fact, that $(1-\frac{y}{n})_+^{n-1}$ converges in $[0,\infty)$ uniformly to e^{-y}, the validity of (1) follows. Putting $t=0$, we see that μ is in fact a probability measure.

Inversion formula and uniqueness theorem

If we assume the uniqueness theorem for the Laplace transform as known, it becomes clear that passing to a subsequence in the above proof was in fact unnecessary. We thus get as a corollary the

Theorem. *[Post-Widder inversion formula.] If φ is the Laplace transform of the probability measure μ on $[0,\infty)$, then for every positive integer the function f_n defined by (4) is a probability density on $[0,\infty)$ and the sequence of the corresponding probability measures converges in distribution to μ.*

On the other hand, it is easy to prove the Post-Widder inversion formula directly in order to get the uniqueness theorem[1]:

$$f_n(x) = \frac{(-1)^n}{n!}\left(\frac{n}{x}\right)^{n+1} \int_0^\infty (-1)^n y^n e^{-\frac{n}{x}y}\, d\mu(y),$$

and we thus get for the cumulative distribution function F_n corresponding to f_n

$$F_n(A) = \int_0^A f_n(x)\, dx = \int_0^\infty \int_{1/A}^\infty \frac{(ny)^n}{(n-1)!} x^{n-1} e^{-nyx}\, dx\, d\mu(y).$$

Now the inner integral in the above formula gives the probability that a random variable with a Gamma distribution with expectation $\frac{1}{y}$ and variance $\frac{1}{ny^2}$ takes a value $\geq \frac{1}{A}$ (see [**5**, p. 47]). If $A \neq y$, this probability converges for $n \to \infty$ to $I(y < A)$. This implies the convergence of $F_n(A)$ to $F(A)$ if F is the distribution function μ and A is a point of continuity of F, and that is the content of the inversion formula.

Finally we should mention that the following seemingly more general version of Bernstein's theorem is in fact a corollary of the one proved above and the uniqueness theorem. For a proof see e.g. [**1**, Corollary 6.14, p. 135].

[1] The following argument is similar to that given in [**2**, pp. 293–294] to prove the uniqueness theorem via the Feller-Dubourdieu inversion formula. See also [**5**, Chapter VII.6].

Theorem. *If $\varphi : (0,\infty) \to \mathbb{R}$ satisfies the condition (ii), then φ is the Laplace transform of a not necessarily finite nonnegative Borel measure on $[0,\infty)$.*

Remarks

1) In a sense, the above proof of Bernstein's theorem is given in [**7**, pp. 139–147]. But our presentation seems to be much more natural and does not presuppose knowledge of the Post-Widder inversion formula, which is the starting point of [**7**].

2) Other classically analytic proofs of Bernstein's theorem are given in [**8**], [**4**] and [**5**]. For modern proofs and generalizations, the reader may consult [**6**], [**3**] and [**1**].

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